KLEENE ALGEBRAS: A CASE-STUDY OF CLONES AND DUALITIES FROM ENDOMORPHISMS

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Abstract. A finite endodualisable algebra is always endoprimal, and this fact has led to the discovery of many endoprimal algebras. Recent investigations by the authors have shown that the finite endoprimal algebras in various well-known quasivarieties of algebras having distributive lattice reducts are exactly the finite endodualisable algebras. In the context of semilattices, B.A. Davey and J.G. Pitkethly found examples of finite algebras which are endoprimal but non-endodualisable. The distinction between entailment in the clone-theoretic and duality-theoretic senses was first revealed in the variety $\mathcal{K}$ of Kleene algebras. This paper considers the subquasivariety $\mathcal{L}$ of $\mathcal{K}$ generated by the four-element chain; this contains only fixpoint-free Kleene algebras. The classes of finite endoprimal and endodualisable algebras in $\mathcal{L}$ are found. These do not coincide and the way in which this happens leads to a better understanding of the relationship between the two entailment concepts.

1. Introduction

This paper has dual objectives. The first, and primary, one is to investigate the distinction between the notion of entailment as it is defined in clone theory and in duality theory. The second, related, objective is to pursue the study of endoprimal and endodualisable algebras by exploring these concepts within the variety of Kleene algebras, a variety which has been extensively investigated from both a logical and an algebraic viewpoint, and which has proved a valuable test-case example in universal algebra and in duality theory.

We shall assume familiarity with the theory of natural dualities. A full account can be found in the monograph by D.M. Clark and B.A. Davey [4]. An introductory survey appears in [7]. An outline of the facts we need, excluding discussion of entailment, can be found in [16], our parallel paper concerning double Stone algebras.

Let $\mathbf{M} = (M; F)$ be any algebra. The algebra $\mathbf{M}$ is called k-**endoprimal** ($k \geqslant 1$) if every k-ary $\operatorname{End}\mathbf{M}$-preserving function on $\mathbf{M}$ is a term function of $\mathbf{M}$. Algebras which are k-endoprimal for every $k \geqslant 1$ are called **endoprimal**. A finite algebra $\mathbf{M}$ is **endodualisable** if $\operatorname{End}\mathbf{M}$ yields a duality on the quasivariety $\mathbb{ISP}(\mathbf{M})$, in the sense defined in [4]. Links between these concepts are summarised in the following basic result, see [4], Proposition 2.2.3.

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PROPOSITION 1.1. *Let* $\mathbf{M}$ *be a finite algebra and let* $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$. *Then* $\mathbf{M}$ *is endoprimal if and only if* $\operatorname{End}\mathbf{M}$ *yields a duality on the free algebras* $\mathbf{F}_{\mathcal{A}}(k)$ *for all* $k \geqslant 1$. *More specifically,* $\mathbf{M}$ *is not k-endoprimal if and only if* $\operatorname{End}\mathbf{M}$ *does not yield a duality on* $\mathbf{F}_{\mathcal{A}}(k)$.

It has been shown that in many quasivarieties a finite algebra is endoprimal if and only if it is endodualisable (see [10], [16] and the papers cited therein). The first, and hitherto only, examples of finite algebras which are endoprimal but not endodualisable were found by B.A. Davey and J.G. Pitkethly [10], among algebras with a semilattice reduct. These examples were re-examined in [15], in the light of the Retraction Test Algebra Lemma obtained there.

Our principal theorems are Theorems 5.1 and 5.2. Together they reveal that there is a plentiful supply of finite Kleene algebras which are endoprimal but not endodualisable. Kleene algebras were already known to illustrate the distinction between entailment in the clone sense and in the duality sense; see [7], p. 87, [13], Section 5, [4], pp. 272–273. Here we show that the relationship between the two entailment concepts lies at the heart of the relationship between endoprimality and endodualisability. This is nicely illustrated by the Kleene algebra examples which led us to the general results.

A Kleene algebra is an algebra $\mathbf{A} = (A; \wedge, \vee, \neg, 0, 1)$ of type $(2, 2, 1, 0, 0)$ in which the reduct $(A; \wedge, \vee, 0, 1)$ is a $\{0, 1\}$-distributive lattice and the unary operation $\neg$, modelling a non-classical negation, satisfies the following laws

(O) $$\neg(a \wedge b) \approx \neg a \vee \neg b, \quad \neg(a \vee b) \approx \neg a \wedge \neg b, \quad \neg 0 \approx 1, \quad \neg 1 \approx 0,$$

(M) $$\neg^2 a \approx a,$$

(K) $$a \wedge \neg a \wedge (b \vee \neg b) \approx a \wedge \neg a.$$

The laws (O) make $\mathbf{A}$ an Ockham algebra, (M) restricts this to be a de Morgan algebra, and the law (K) defines Kleene algebras within the variety of de Morgan algebras. Any finite chain $\mathbf{n} = \{0, 1, \dots, n-1\}$ can be endowed with a unique Kleene negation: $\neg k = n - 1 - k$. It was shown by J.A. Kalman [17] that the chains $\mathbf{2}$ and $\mathbf{3}$ are the only non-trivial subdirectly irreducible Kleene algebras; $\mathcal{K} = \mathbb{ISP}(\mathbf{3})$ and $\mathbb{ISP}(\mathbf{2}) = \mathcal{B}$, the variety of Boolean algebras. Any Boolean algebra is endodualisable and hence endoprimal [8]. In the opposite direction it is easy to exhibit Kleene algebras which fail even to be 1-endoprimal: it is easy to see that in a non-Boolean Kleene algebra $\mathbf{M}$ with a $\neg$-fixpoint c, the constant map from M onto c witnesses the failure of 1-endoprimality. The simplest example of a non-Boolean fixpoint-free Kleene algebra is the four-element chain $\mathbf{4}$. In this paper we characterise the endodualisable and endoprimal finite algebras in $\mathcal{L} := \mathbb{ISP}(\mathbf{4})$. Besides nicely illustrating the entailment ideas that we explore, this analysis provides a new example of optimal natural dualities which is sufficiently rich to exhibit interesting features yet simple enough to be presented quite briefly.

We remark that different subquasivarieties of a quasivariety may be expected to behave in different ways as regards endodualisability (cf. [16], in which the variety of double Stone algebras is studied). M.E. Adams and W. Dziobiak in [1] prove that $\mathcal{K}$ has uncountably many subquasivarieties; their proof implies that there is a countably infinite family of finitely generated subquasivarieties. Therefore we may expect it to be a major task to complete the characterisation of endodualisable and endoprimal fixpoint-free Kleene algebras outside $\mathcal{L}$, and we do not embark upon it in this paper. However we note that several of the techniques we employ below can be formulated so that they are applicable in the context of algebras satisfying conditions not specific to $\mathcal{L}$, or even to Kleene algebras. These generalities, and further investigation of $\mathcal{K}$ based on them, will be pursued elsewhere.

2. Entailment by endomorphisms

Throughout our discussion of generalities we work within the same framework as in [4]. Our focus is on algebras which are known to be dualisable. In our Kleene algebra application we exploit the fact that any finite Kleene algebra is dualised by $\mathbb{S}(\mathbf{M}^2)$, since $\mathbf{M}$ has a 3-ary NU term, because it has a lattice reduct.

We take a finite algebra $\mathbf{M}$ and consider a candidate alter ego $\underset{\sim}{\mathbf{M}} = (M; G, H, R, \tau)$ for $\mathbf{M}$. Here, as usual, G, H and R are, respectively, sets (possibly empty) of operations, partial operations and relations which are algebraic over $\mathbf{M}$, meaning that the relations and the graphs of the (partial) operations are subalgebras of finite powers of $\mathbf{M}$, while τ denotes the discrete topology. There is then a dual adjunction between $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ and $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ set up by hom-functors D and E into $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$, respectively. As always, subsets of powers of M, on both the algebraic and dual sides, are structured pointwise from M. In particular, $r^{D(\mathbf{A})}$ denotes the algebraic relation r acting on $D(\mathbf{A})$, so that, if r is n-ary,

$$(x_1, \ldots, x_n) \in r^{D(\mathbf{A})} \Leftrightarrow (\forall a \in A)\, (x_1(a), \ldots, x_n(a)) \in r.$$

We can, and usually do, replace an n-ary operation g by its graph; we denote this $(n+1)$-ary algebraic relation by graph(g).

We recall that $G \cup H \cup R$ **entails the algebraic relation** s **on** $D(\mathbf{A})$ if every $\mathcal{X}$-morphism $u : D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ preserves s. If $G \cup H \cup R$ entails s on $D(\mathbf{A})$ for every $\mathbf{A} \in \mathcal{A}$, we say that $G \cup H \cup R$ **entails** s and write $G \cup H \cup R \vdash s$. For relations r and s we write as usual $r \vdash s$ in place of $\{r\} \vdash s$.

If $G \cup H \cup R$ dualises $\mathbf{M}$, then any relation r in $G \cup H \cup R$ which is entailed by $G \cup H \cup (R \setminus \{r\})$ can be deleted without destroying the duality. The significance of this for the study of endodualisability is clear: $\mathbf{M}$ is endodualisable if and only if $\operatorname{End} M$ entails every element of a set known to yield a duality on $\mathbb{ISP}(\mathbf{M})$. Central to the identification of entailed relations is the Test Algebra Lemma. This appeared initially in [12], Section 2, and is formulated in entailment terms in [13], Lemma 2.3, and [4], Lemma 8.1.3. We reproduce this statement here for reference. Here (and likewise below) $\mathbf{s}$ denotes the algebraic relation s regarded as an algebra in $\mathcal{A}$.

Lemma 2.1 (Test Algebra Lemma). *Let $\mathbf{M}$ be a finite algebra, let G, H, R be as above and let s be an algebraic relation. Then the following are equivalent:*

(a) *$G \cup H \cup R$ entails s;*
(b) *$G \cup H \cup R$ entails s on $D(\mathbf{s})$.*

Moreover, $G \cup H \cup R$ entails s whenever $G \cup H \cup R$ yields a duality on $\mathbf{s}$.

We shall use the term **test algebra** for an algebra $\mathbf{A} \in \mathbb{ISP}(\mathbf{M})$ witnessing the failure of the structure $\underset{\sim}{\mathbf{M}}$ to yield a duality on $\mathbb{ISP}(\mathbf{M})$. In particular, we use it for an algebra $\mathbf{A}$ witnessing non-endodualisability of $\mathbf{M}$.

The theorem stated below is an amalgam of results proved and used in earlier investigations. Part (1) is a special case of the Retraction Test Algebra Lemma ([15], Lemma 4). Part (2) goes back to Davey and Pitkethly, [10], Theorems 1.5 and 1.2. Part (3) was proved in [16], Theorem 3.6.

THEOREM 2.2. *Let* $\mathbf{M}$ *be a finite algebra.*

(1) *If* $\mathbf{M}$ *is not endodualisable, and this is witnessed by the failure of* $\operatorname{End}\mathbf{M}$ *to yield a duality on a test algebra* $\mathbf{s} \in \mathbb{ISP}(\mathbf{M})$ *such that* $\mathbf{s}$ *is a retract of a free algebra* $\mathbf{t} := \mathbf{F}_{\mathbb{ISP}(\mathbf{M})}(k)$*, then* $\mathbf{t}$ *also witnesses the fact that* $\mathbf{M}$ *is not endodualisable, and consequently* $\mathbf{M}$ *is not* k*-endoprimal.*
(2) *Assume that* $\mathbf{D}$ *is a retract of* $\mathbf{M}$*. Then if* $\mathbf{D}$ *is endodualisable* (k*-endoprimal, endoprimal*) *so is* $\mathbf{M}$*. This is true in particular if* $\mathbf{M} = \mathbf{D}^n$ *for some* $n > 1$*.*
(3) *Let* $n > 1$*. Assume that every subalgebra of* $\mathbf{D}$ *is indecomposable and that every homomorphism from* $\mathbf{D}^n$ *to* $\mathbf{D}$ *is expressible in the form* $e \circ \pi_i$*, for some projection map* $\pi_i : \mathbf{D}^n \to \mathbf{D}$ *and some* $e \in \operatorname{End}\mathbf{D}$*. Then if* $\mathbf{D}$ *is not endodualisable* (*not* k*-endoprimal, not endoprimal*) *then neither is* $\mathbf{D}^n$*.*

Part (3) in Theorem 2.2 provides a partial converse to part (2). Even under the restrictions on $\mathbf{D}$ given in (3), the full converse of (2) is not true in general, for reasons that Theorem 2.3 reveals.

Suppose we seek to discover whether a given finite algebra $\mathbf{M}$ is endodualisable where $\mathbf{M}$ retracts onto some non-endodualisable algebra $\mathbf{D}$ for which a dualising structure $\underset{\sim}{\mathbf{D}}$ is already known. We can 'lift up' the structure $\underset{\sim}{\mathbf{D}}$ to find a dualising structure $\underset{\sim}{\mathbf{M}}$. This method, which was first employed by B.A. Davey in [8], is made precise in the following theorem; see [10], Theorems 1.5 and 1.6 and [15], Theorems 3 and 4. The final statement in the theorem relies on the fact that $\operatorname{End}\mathbf{M}$ yields a duality on any algebra (isomorphic to) a retract of $\mathbf{M}$.

To state the theorem we introduce some notation. Let $\mathbf{D}$ be a subretract of a finite algebra $\mathbf{M}$. Let s be an n-ary algebraic relation on D. Then $\mathbf{s} \leqslant \mathbf{D}^n$, and $\mathbf{D}^n$ may be considered as a subalgebra of $\mathbf{M}^n$. Combining the two inclusions we may regard $\mathbf{s}$ as a subalgebra of $\mathbf{M}^n$; when this is done we write s_D in place of s, as a reminder of where this algebraic relation came from. Given a set of relations S on D we let $S_D := \{s_D \mid s \in S\}$.

THEOREM 2.3. *Let* $\mathbf{D}$ *be a finite algebra. Let* $\mathbf{M}$ *be a finite algebra in* $\mathbb{ISP}(\mathbf{D})$ *and assume that* $\mathbf{D}$ *is a subretract of* $\mathbf{M}$*. If* $\mathbf{D}$ *is dualised via* $\underset{\sim}{\mathbf{D}} = (D; \operatorname{End}\mathbf{D}, S, \tau)$*, with each* $s \in S$ *a finitary algebraic relation on* $\mathbf{D}$*, then* $\mathbf{M}$ *is dualised via* $\underset{\sim}{\mathbf{M}} = (M; \operatorname{End}\mathbf{M}, S_D, \tau)$*. If, further,* $\mathbf{s}$ *is* (*isomorphic to*) *a retract of* $\mathbf{M}$ *for each* $s \in S$ *then* $\mathbf{M}$ *is endodualisable.*

The theoretical developments that followed the introduction of the notion of a test algebra culminated in the solution of the Entailment Problem [9]. The solution, which is closely bound up with the ideas underlying the Test Algebra Lemma, gives both semantic and syntactic descriptions of the relations entailed by a given set $G \cup H \cup R$. The long-standing Entailment Problem in duality theory ([7], Problem 1) sought a list of constructs which, applied to $G \cup H \cup R$, yields every algebraic relation entailed by this set in a finite number of steps. One such complete set consists of term manipulation, intersection and retractive projection; see [4], 9.2.6. The relationship between retractive projection and homomorphic relational product (which we need below) is given in Proposition 3.9 of [9]. In important special cases neither of these two constructs, nor term manipulation in full generality, is required and a set of more convenient constructs suffices: a list, with redundancies, is given in [4], 2.4.5. This simplification occurs in particular (see [4], 9.3.3) when $G \cup H = \varnothing$ and, for all $n \in \mathbb{N}$, every n-ary algebraic relation s satisfies:

(H) every homomorphism $h\colon \mathbf{s} \to \mathbf{M}$ is essentially unary (that is, $h = e \circ \rho_i$ for some i and some partial endomorphism e of $\mathbf{M}$ with $\operatorname{dom} e \subseteq \operatorname{Im} \rho_i$, where $\rho_i := \pi_i{\restriction_s}$ is the restriction of the natural projection $\pi_i\colon \mathbf{M}^n \to \mathbf{M}$).

The condition (H) is satisfied for every s when $\mathbf{M}$ generates a congruence-distributive variety and is such that every subalgebra of $\mathbf{M}$ is subdirectly irreducible (see [9], 4.2 or [4], 8.5.5). When working with arbitrary finite algebras $\mathbf{M}$ drawn from a quasivariety $\mathbb{ISP}(\mathbf{D})$, condition (H) is seldom satisfied for all s. This failure brings homomorphic relational products into play (for an illustration, see the proof of Proposition 4.4 below). We stress that relational product in general is not an admissible construct for duality entailment; see the Kleene algebra illustration on pages 272–273 of [4].

Let S be a family of relations on a set M. Then a relation r on M is **clone-entailed** by S on M if, for all $k \geqslant 1$, any function $f\colon M^k \to M$ which preserves all relations in S also preserves r, that is, r is a subalgebra of $\mathbf{M}^m$ (m being the arity of r), where $\mathbf{M} = (M; \operatorname{Pol}(S))$. The set of all such relations r is usually denoted by $\operatorname{Inv}(\operatorname{Pol}(S))$. It is well known that the relations in $\operatorname{Inv}(\operatorname{Pol}(S))$ can be described either syntactically or semantically. The semantic description says that r is clone-entailed by S if and only if r can be built from S using a finite number of applications of product, intersection, trivial relations, repetition removal and projection; see [2] and [3], [18], Chapter 2, or Exercises 9.3–9.6 in [4]. By contrast with duality-entailment, a pair of algebraic relations r, s always clone-entails the relational product $r \cdot s$.

The relationship between (duality) entailment and clone-entailment is not fully understood. It is known that it is possible for $G \cup H \cup R$ to clone-entail every finite algebraic relation on $\mathbf{M}$ but to fail to dualise $\mathbf{M}$, but the circumstances under which this phenomenon occurs, and what it signifies, are still obscure. In particular, we may ask what it means for $\mathbf{M}$ to be endoprimal but not endodualisable. More explicitly, we may ask what it means for some finitary algebraic relation r on $\mathbf{M}$ to be clone-entailed but not entailed by (the graphs of) the endomorphisms of $\mathbf{M}$. From a semantic viewpoint, a clear difference can be seen: clone-entailment allows all relational products, whereas (duality) entailment allows only homomorphic relational products. Thus we may expect relational products appearing in the construction of r from the endomorphisms of $\mathbf{M}$ to be non-homomorphic relational products. Exactly how this behaviour happens in general is not clear. However it is strikingly illustrated by the results presented below which characterise algebras $\mathbf{M}$ in $\mathbb{ISP}(\mathbf{4})$ which are endoprimal but not endodualisable. Our work is therefore a step along the road to a better understanding of the distinction between the two entailment notions.

We proceed to give sufficient conditions for endoprimality, paralleling that for endodualisability in Theorem 2.3.

LEMMA 2.4. *Let* $\mathbf{D}$ *be a subretract, with* $\gamma\colon \mathbf{M} \twoheadrightarrow \mathbf{D}$ *as the retraction, of an algebra* $\mathbf{M}$*. Assume that* S *is a set of algebraic relations on* D *which clone-entails an algebraic relation* r *on* D*. Then the set* $S_D \cup \{\gamma\}$ *clone-entails* r_D *on* M*.*

Proof. Let $f\colon M^n \to M$ preserve s_D for each $s \in S$ and also preserve the map $\gamma\colon \mathbf{M} \twoheadrightarrow \mathbf{D} \leqslant \mathbf{M}$, regarded as an endomorphism of $\mathbf{M}$. Because f preserves γ it follows that f preserves $\mathrm{fix}(\gamma)$. Thus $f' := f{\restriction}_{\mathbf{D}^n}\colon D^n \to D$ is well defined. As f preserves s_D for each $s \in S$ it follows that f' preserves s for all $s \in S$. Since S clone-entails r on D, we deduce that f preserves r_D since f' does. □

PROPOSITION 2.5. *Let* $\underset{\sim}{\mathbf{D}} = (D; R, \tau)$ *dualise* $\mathbf{D}$*. Let* $\mathbf{M} \in \mathbb{ISP}(\mathbf{D})$ *be such that* $\mathbf{D}$ *is a subretract of* $\mathbf{M}$*. Assume that there is a subset* S *of* R *such that*

(i) $\mathrm{End}\,\mathbf{M}$ *entails* s_D *for each* $s \in S$*, and*
(ii) S *clone-entails* r *on* D *for all* $r \in R \setminus S$*.*

Then $\mathbf{M}$ *is endoprimal.*

Proof. By Theorem 2.3, $\underset{\sim}{\mathbf{M}} = (M; \mathrm{End}\,\mathbf{M}, R_D, \tau)$ dualises $\mathbf{M}$. Let $f\colon M^n \to M$ preserve $\mathrm{End}\,\mathbf{M}$. Then, since $\mathrm{End}\,\mathbf{M}$ entails s_D on the dual $D(\mathbf{F}_{\mathbb{ISP}(\mathbf{M})})(n)$ of the free algebra on n generators in $\mathbb{ISP}(\mathbf{M})$ for each $s \in S$ and this dual can be identified with M^n (see [4], 2.2.1), we have that f preserves s_D for each $s \in S$.

We now require that f preserves r_D for each relation r on D which is clone-entailed on D by S. As in the proof of Lemma 2.4, we may regard the retraction γ as an element of $\mathrm{End}\,\mathbf{M}$, and deduce that it is preserved by f. Therefore, by the lemma, f preserves r_D for each $r \in R \setminus S$. Hence f preserves every relation r_D for $r \in R$. Therefore f is a term function as $\mathrm{End}\,\mathbf{M} \cup R_D$ dualises $\mathbf{M}$. □

PROPOSITION 2.6. *Let* $\underset{\sim}{\mathbf{D}} = (D; \mathrm{End}\,\mathbf{D}, R, \tau)$ *dualise the finite algebra* $\mathbf{D}$*. Let* $\mathbf{M} \in \mathbb{ISP}(\mathbf{D})$ *and assume* $\mathbf{D}$ *is a retract of* $\mathbf{M}$*. Assume that there is a subset* S *of* R *such that*

(i) $\mathbf{s}$ *is a retract of* $\mathbf{M}$ *for all* $s \in S$*, and*
(ii) r *is a relational product of relations from* S *for all* $r \in R \setminus S$*.*

Then $\mathbf{M}$ *is endoprimal.*

Proof. As noted above, $\mathrm{End}\,\mathbf{M}$ yields a duality on every retract of $\mathbf{M}$. Hence, by 2.1, $\mathrm{End}\,\mathbf{M}$ entails every relation s_D such that $s \in S$. Condition (ii) tells us that S clone-entails r for every $r \in R \setminus S$. The endoprimality of $\mathbf{M}$ now follows from Proposition 2.5. □

3. KLEENE ALGEBRAS: PRELIMINARIES

For simplicity of notation we shall use $\mathbf{A}$ to denote both a Kleene algebra and its $\{0,1\}$-distributive lattice reduct; which is meant will be clear from the context.

We shall make free use of Priestley duality for distributive lattices and of the associated duality for Kleene algebras, originally due to W.H. Cornish and P.R. Fowler [5]. We summarise here only the minimum needed to establish notation. Because all the algebras we deal with are finite, the duality for distributive lattices is purely order-theoretic, built on Birkhoff's classic representation; the topology that is needed in the representation of arbitrary algebras can be suppressed. We always identify a finite distributive lattice $\mathbf{A}$ with the lattice of upsets of a finite ordered set; this ordered set is simply the join-irreducible elements of $\mathbf{A}$ with the reverse of the induced order.

In this paper a (finite) **Kleene space** will mean a structure $(Y; \leqslant, g)$, where $(Y; \leqslant)$ is a finite ordered set and $g\colon Y \to Y$ is an order-reversing involution such that y and $g(y)$ are comparable for all $y \in Y$. The associated morphisms are order-preserving maps which commute with the g-map. We denote the resulting category by $\mathcal{Y}$ and by H and K the functors setting up the contravariant equivalence between the finite algebras in $\mathcal{K}$ and $\mathcal{Y}$. The identification mentioned above is just the identification of the lattice $\mathbf{A}$ with $KH(\mathbf{A})$; the negation is captured by $\neg a = Y \setminus g^{-1}(a)$ $(a \in \mathbf{A})$, where $Y = H(\mathbf{A})$. A $\mathcal{K}$-homomorphism $f\colon \mathbf{A} \to \mathbf{B}$ is related to its dual $\varphi := H(f)\colon H(\mathbf{B}) \to H(\mathbf{A})$ by $f = \varphi^{-1}$.

In [16] we exploited the fact that properties of a double Stone algebra are derived, in some measure, from properties of its core. Analogously, given a Kleene algebra $\mathbf{A}$, we may hope to find an associated algebraic structure which is simpler than $\mathbf{A}$ and whose form determines whether or not $\mathbf{A}$ is endoprimal, and so on. For an algebra $\mathbf{A} \in \mathcal{K}$ the candidate for this simpler structure is the ideal $\mathbf{A}^{\wedge} := \{\, a \wedge \neg a \mid a \in A \,\}$. There is, of course, a filter defined in like manner by $\mathbf{A}^{\vee} := \{\, a \vee \neg a \mid a \in A \,\}$, obtained from $\mathbf{A}^{\wedge}$ by applying $\neg$ elementwise. Although the lattices $\mathbf{A}^{\wedge}$ and $\mathbf{A}^{\vee}$ do not alone carry enough information for our purposes, they do arise repeatedly in our arguments. On the dual side, we view a Kleene space Y as the union of three disjoint sets:

$$Y^{\wedge} := \{\, y \in Y \mid y > g(y) \,\},\ Y^{\vee} := \{\, y \in Y \mid y < g(y) \,\},\ Y^{\circ} := \{\, y \in Y \mid y = g(y) \,\}.$$

If $\mathbf{A}$ be a finite Kleene algebra, regarded as the lattice of upsets of its dual space $Y = H(\mathbf{A})$, then it is elementary to see that

$$\mathbf{A}^{\wedge} = \{\, a \in A \mid a \subseteq Y^{\wedge} \,\},\ \mathbf{A}^{\vee} = \{\, a \in A \mid a \supseteq (Y^{\wedge} \cup Y^{\circ}) \,\}$$

and that $Y^{\wedge}$ is the dual space of the lattice $\mathbf{A}^{\wedge}$. Thus, $\mathbf{A}^{\wedge}$ is Boolean if and only if $Y^{\wedge}$ is an antichain. Below, the g-fixpoints in Kleene space diagrams are shown shaded.

We now fix some notation concerning the chain $\mathbf{4}$. We denote the underlying set of $\mathbf{4}$ by $4 = \{0, a, b, 1\}$, where $0 < a < b\,(= \neg a) < 1$. The dual $H(\mathbf{4})$ is a three-element chain $\lambda < \nu < \mu$, with $\nu = g(\nu)$, $g(\mu) = \lambda$ and $g(\lambda) = \mu$. We need to characterise those Kleene algebras which lie in $\mathcal{L} := \mathbb{ISP}(\mathbf{4})$.

Lemma 3.1. *Let $\mathbf{A}$ be a non-trivial finite Kleene algebra, with dual space $Y = H(\mathbf{A})$. Then the following are equivalent:*

(a) $\mathbf{A} \in \mathbb{ISP}(\mathbf{4})$;
(b) $Y = \bigcup\{\, \operatorname{Im} H(x) \mid x \in \hom(\mathbf{A}, \mathbf{4}) \,\}$;
(c) *every point of Y is comparable to a g-fixpoint, that is, $Y = {\uparrow}Y^{\circ} \cup {\downarrow}Y^{\circ}$.*

Proof. For the equivalence of (a) and (b) we use Theorem 7.4.1(vi) of [4]. The finite case of this, which is all we need here, appears, with its proof, in [16] (Lemma 2.4).

We claim that (b) and (c) are equivalent. For each $x \in \mathrm{hom}(\mathbf{A}, \mathbf{4})$, the image of the map $H(x)\colon H(\mathbf{4}) \to H(\mathbf{A})$ is a chain, with the image of the g-fixpoint ν a g-fixpoint in $H(\mathbf{A})$ and the images of λ and $\mu = g(\lambda)$ being g-images of one another. The equivalence of (b) and (c) now follows easily. □

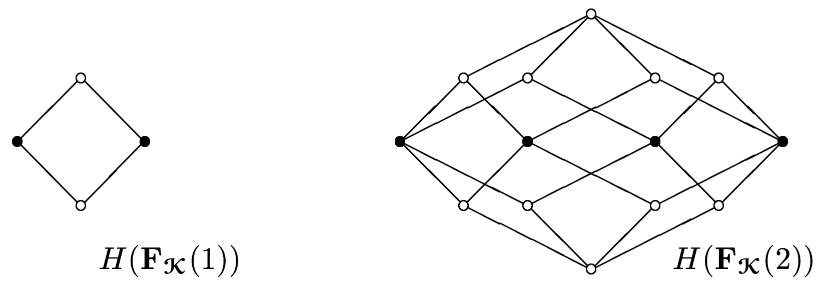


FIGURE 1

The free Kleene algebras $\mathbf{F}_{\mathcal{K}}(k)$ $(1 \leqslant k < \omega)$ have been described in many ways. For our purposes, we need only some facts about their Priestley duals. The algebra $\mathbf{f}$ in Table 1 is $\mathbf{F}_{\mathcal{K}}(1)$. For future reference, its dual $H(\mathbf{F}_{\mathcal{K}}(1))$ and that of $\mathbf{F}_{\mathcal{K}}(2)$ are shown in Figure 1. The procedure for obtaining $Y_k := H(\mathbf{F}_{\mathcal{K}}(k))$ in general can be found in [6] and, by natural duality methods, in [11]. The following lemma is immediate, from the way that $H(\mathbf{F}_{\mathcal{K}}(k))$ is constructed.

LEMMA 3.2. *Let* $1 \leqslant k < \omega$ *and let* $Y_k := H(\mathbf{F}_{\mathcal{K}}(k))$. *If* $u, v \in {Y_k}^{\wedge}$ *with* $u \geqslant g(v)$, *then there exists a* g*-fixpoint* w *with* $u \geqslant w \geqslant g(v)$. *In particular, each point of* Y_k *is comparable to a* g*-fixpoint.*

Consequently, by Lemma 3.1, all the algebras $\mathbf{F}_{\mathcal{K}}(k)$ lie in $\mathcal{L}$. Accordingly, $\mathbf{F}_{\mathcal{L}}(k) = \mathbf{F}_{\mathcal{K}}(k)$ for each finite k.

As indicated in Sections 1 and 2, retracts play an important role in characterisations of endodualisable and endoprimal algebras.

LEMMA 3.3. *Let* $\mathbf{A}$ *be a finite Kleene algebra in* $\mathbb{ISP}(\mathbf{4})$. *Then*

(1) $\mathbf{A}$ *has* $\mathbf{4}$ *as a retract if and only if* $\mathbf{A}$ *is not Boolean.*
(2) $\mathbf{A}$ *has* $\mathbf{6}$ *as a retract if and only if* $\mathbf{A}^{\wedge}$ *is not Boolean.*
(3) $\mathbf{A}$ *has* $\mathbf{F}_{\mathcal{K}}(1)$ *as a retract if and only if* $H(\mathbf{A})$ *has a pair of distinct* g*-fixpoints which have a common upper bound (and common lower bound).*

Proof. The proofs of (1) and (2) are elementary, and we omit them. We now prove (3). The given condition on g-fixpoints is just that for $H(\mathbf{F}_{\mathcal{K}}(1))$ to embed in $H(A)$ (with the embedding a Kleene space morphism). It is exactly the condition for $\mathbf{A}$ to have $\mathbf{F}_{\mathcal{K}}(1)$ as a homomorphic image, and hence also as a retract because the latter algebra is free. □

We remark that if $\mathbf{A}$ is a finite non-Boolean member of $\mathcal{L}$, with dual space $Y = H(\mathbf{A})$ then the algebra $\mathbf{4}$ may be regarded as a subretract of $\mathbf{A}$: the elements a and b $(= \neg a)$ of $\mathbf{4}$ are taken to be, respectively, the top element of $\mathbf{A}^{\wedge}$ (that is, the upset $Y^{\wedge}$) and the bottom element of $\mathbf{A}^{\vee}$ (that is, the upset $Y^{\wedge} \cup Y^{\circ}$).

LEMMA 3.4. *Let* $\mathbf{A}$ *be a finite algebra in* $\mathcal{K}$ *(in* $\mathcal{L}$*). Then the following are equivalent:*

(a) $\mathbf{A}$ *is indecomposable in* $\mathcal{K}$ *(in* $\mathcal{L}$*);*
(b) *the lattice reduct of* $\mathbf{A}$ *is indecomposable;*
(c) $H(\mathbf{A})$ *is connected as an ordered set.*

Proof. First consider $\mathbf{A} \in \mathcal{K}$. Clearly (b) implies (a). Under Priestley duality, products are transformed into disjoint unions. The equivalence of (b) and (c) follows from this, and the contrapositive of (a) implies (b) comes from noting that the order components of $H(\mathbf{A})$ are closed under the g-action, so that the duals of the co-ordinate projections onto the factors of a lattice product are Kleene space morphisms.

For the corresponding result for $\mathcal{L}$ we simply note in addition that each order component of the dual space $H(\mathbf{A})$ is the dual space of an algebra in $\mathcal{L}$, every point being comparable to a g-fixpoint. □

The following easy lemma is central to our later investigations, as it enables us to describe the endomorphisms of any finite Kleene algebra and also the hom-sets which arise in connection with natural dualities. The conclusions of the lemma hold true when the variety $\mathcal{K}$ is replaced by a variety $\mathcal{A}$ satisfying suitable general conditions, and with the algebras involved not restricted to be finite. In particular, it is sufficient to assume that $\mathcal{A}$ has factorisable congruences and that every subalgebra of an indecomposable algebra in $\mathcal{A}$ is also indecomposable. These conditions hold in particular when $\mathcal{A}$ is the subquasivariety $\mathbb{ISP}(\mathbf{4})$ of Kleene algebras.

LEMMA 3.5. *Let* $\mathbf{L}$, $\mathbf{M}$ *be finite Kleene algebras and let* $e \in \hom(\mathbf{L}, \mathbf{M})$. *Assume that* $\mathbf{L} = \mathbf{L}_1 \times \cdots \times \mathbf{L}_p$, *where* $\mathbf{L}_1, \dots, \mathbf{L}_p$ *are indecomposable.*

(i) *Assume that* $\mathbf{M}$ *is indecomposable. Then there exist* j *and a homomorphism* $g\colon \mathbf{L}_j \to \mathbf{M}$ *such that* $e = g \circ \pi_j$, *where* $\pi_j\colon \mathbf{L} \to \mathbf{L}_j$ *is the canonical projection.*
(ii) *Assume that* $\mathbf{M} = \mathbf{M}_1 \times \cdots \times \mathbf{M}_q$, *where* $\mathbf{M}_1, \dots, \mathbf{M}_q$ *are indecomposable. Then there exist a map* $\varepsilon\colon \{1, \dots, q\} \to \{1, \dots, p\}$ *and maps* $e_i \in \hom(\mathbf{L}_{\varepsilon(i)}, \mathbf{M}_i)$ $(i = 1, \dots, q)$ *such that*

$$e((a_1, \dots, a_p)) = (e_1(a_{\varepsilon(1)}), \dots, e_q(a_{\varepsilon(q)})) \quad \text{for all } (a_1, \dots, a_p) \in \mathbf{L}.$$

Proof. The proof, which makes use of Lemma 3.4, proceeds in exactly the same way as the proof of the analogous result in [16] (Lemma 2.1). □

4. NATURAL DUALITIES FOR $\mathcal{L}$

The natural duality for $\mathcal{K}$ was first described in [14]. It was subsequently shown in [11] to arise as a generalised piggyback duality. For $\mathcal{L}$ a generalised piggyback duality may likewise be derived by appealing to the Piggyback Duality Theorem for distributive-lattice-based algebras (see for example [7], 5.1, or [4], 7.2.1). It tells us that $\underset{\sim}{\mathbf{4}} = (4; e, m, \ell, d)$ dualises $\mathbf{4}$, where the endomorphism e maps a to 0 and b to 1, and the piggyback subalgebras of $\mathbf{4}^2$ are $\mathbf{m} = \mathbf{4}^2 \setminus \{(0,1),(1,0)\}$, the subalgebra $\boldsymbol{\ell}$ listed in Table 1 and $\mathbf{d} := \{(0,0),(1,1)\}$. For future use the Priestley duals of $\mathbf{m}$ and $\boldsymbol{\ell}$ are shown in Figure 2.

In order to understand fully the finite endoprimal algebras in $\mathcal{L}$ we need to work not just with the subalgebras thrown up by the Piggyback Duality Theorem. The

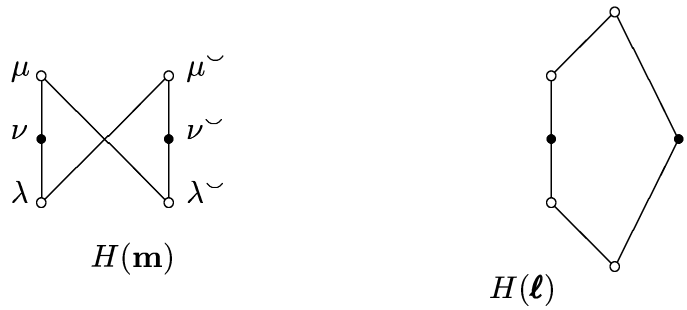


FIGURE 2

NU Duality Theorem (see [4], 2.3.4) ensures that the full set of subalgebras $\mathbb{S}(\mathbf{4}^2)$ of $\mathbf{4}^2$ dualises $\mathbf{4}$. We may expect there to exist minimal dualising sets within $\mathbb{S}(\mathbf{4}^2)$ which are not subsets of $\{e, m, \ell, d\}$. Such minimal dualising sets were introduced and investigated in a general setting in [13] where optimal natural dualities for $\mathcal{K}$ were discussed (see also [4], Section 8.4). We recall here a few basic concepts from [13] (see also [4], pp. 239–240), restricted, for simplicity, to the particular situation that concerns us.

Let $s \in \mathbb{S}(\mathbf{4}^2)$ and let $u : D(\mathbf{s}) \to 4$ be any map. Define

$$U = \mathrm{Fail}_{\mathbf{s}}(u) := \{r \in \mathbb{S}(\mathbf{4}^2) \mid u \text{ fails to preserve } r\}.$$

If $U \neq \varnothing$ we call U a **weak failset of** s (within $\mathbb{S}(\mathbf{4}^2)$), and if $s \in U$ we call U a **failset of** s (within $\mathbb{S}(\mathbf{4}^2)$). We refer to U as a **failset** if it is a failset of some $s \in \mathbb{S}(\mathbf{4}^2)$, and as a **globally minimal failset** (within $\mathbb{S}(\mathbf{4}^2)$) if it is a minimal element (with respect to set inclusion) of the family $\mathcal{F}$ of all failsets (within $\mathbb{S}(\mathbf{4}^2)$). A set $T \subseteq \mathbb{S}(\mathbf{4}^2)$ is called a **transversal** of $\mathcal{F}$ if T intersects each $U \in \mathcal{F}$ but no proper subset of T does. We further say that a relation $s \in \mathbb{S}(\mathbf{4}^2)$ is **absolutely unavoidable** if whenever a subset R of $\mathbb{S}(\mathbf{4}^2)$ dualises $\mathbf{4}$, then R contains s or its converse $s^{\smile}$.

The Optimal Duality Theorem (see [13], 4.4 or [4], 8.3.10) tells us that the minimal dualising sets within $\mathbb{S}(\mathbf{4}^2)$ we are seeking are the transversals of the globally minimal failsets, considered relative to $\mathbb{S}(\mathbf{4}^2)$. We observe that the theory of the structure of globally minimal failsets developed in [19] under the assumption that condition (H) is satisfied is of no assistance to us here. Indeed, our results below indicate that condition (H) is necessary for the structure theorems in [19] to be valid.

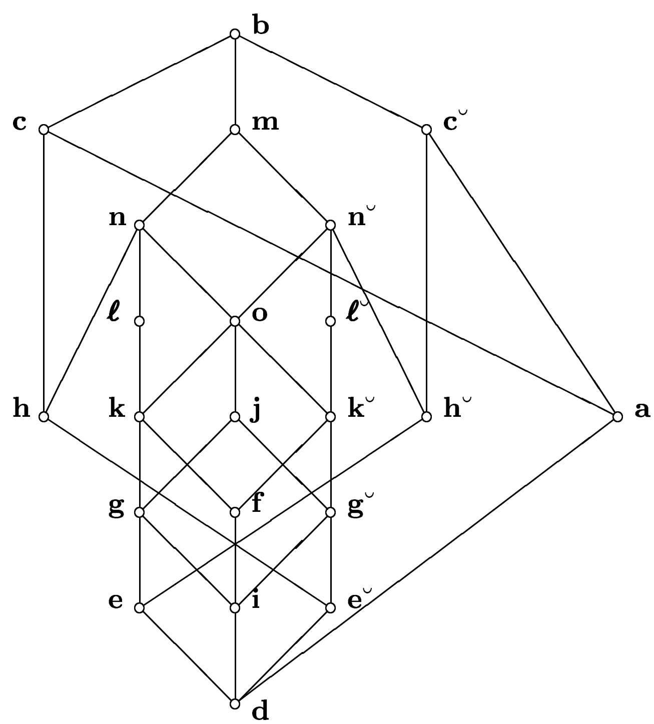
The algebra $\mathbf{4}$ has $\mathbf{2} := \{0, 1\}$ as its only proper subalgebra. The only decomposable subalgebras of $\mathbf{4}^2$ are therefore $\mathbf{a} := \mathbf{2}^2$, $\mathbf{b} := \mathbf{4}^2$, $\mathbf{c} := \mathbf{2} \times \mathbf{4}$ and $\mathbf{c}^{\smile}$. It is an easy exercise to check that there are, up to converses, twelve indecomposable subalgebras. Table 1 lists these. We recall for emphasis that $\mathbf{e} = \mathrm{graph}(e)$. Note also that the diagonal, $\mathbf{i}$, is the graph of the identity map, and that $\mathbf{j}$ is the joint kernel $\ker(e, e)$ (see the list of constructs below).

As the only proper partial endomorphism of $\mathbf{4}$ is the inclusion of $\mathbf{2}$ into $\mathbf{4}$, which can be extended to an endomorphism, we shall not be concerned with partial endomorphisms which are not endomorphisms.

In our analysis of globally minimal failsets for $\mathcal{L}$ we shall use the following entailment constructs on algebraic relations on M: trivial relations, converse, repetition

	d	**e**	**f**	**g**	**h**	**i**	**j**	**k**	**ℓ**	**m**	**n**	**o**
$(0,0)$	•	•	•	•	•	•	•	•	•	•	•	•
$(0,a)$					•		•			•	•	•
$(0,b)$					•					•	•	
$(0,1)$												
$(a,0)$		•		•			•	•	•	•	•	•
(a,a)			•	•		•	•	•	•	•	•	•
(a,b)			•					•	•	•	•	•
$(a,1)$									•	•		
$(b,0)$									•	•		
(b,a)			•					•	•	•	•	•
(b,b)			•	•		•	•	•	•	•	•	•
$(b,1)$		•		•			•	•	•	•	•	•
$(1,0)$												
$(1,a)$					•					•	•	
$(1,b)$					•		•			•	•	•
$(1,1)$	•	•	•	•	•	•	•	•	•	•	•	•

Table 1



$\langle \mathbb{S}(\mathbf{4}^2); \subseteq \rangle$

Figure 3

removal, trivial expansion, intersection, product, joint kernels, action by an endomorphism and homomorphic relational product. We give brief details on the last construct while for the other we refer the reader to [4], pp. 57–59.

Given binary relations r and s, consider the relational product

$$r \cdot s := \{\, (c_1, c_2) \in M^2 \mid (\exists c \in M)(c_1, c) \in r \text{ and } (c, c_2) \in s \,\}.$$

The formation of this relational product is an admissible construct provided it is a **homomorphic relational product**, that is, if there exists a binary homomorphism $F\colon r \cdot s \to \mathbf{M}$ such that $(c_1, F(c_1, c_2)) \in r$ and $(F(c_1, c_2), c_2) \in s$ (see [4], p. 275). Observe that $t := r \cdot s$ is a homomorphic relational product precisely when

$$D(\mathbf{t}) \models (\exists x)(r(\rho_1, x) \wedge s(x, \rho_2)).$$

We remark that, for a endomorphism e and binary relation s, the relational product $(\operatorname{graph}(e)) \cdot s$ is a homomorphic relational product, and coincides with $e \cdot s$.

We recall that a subalgebra $\mathbf{B}$ of an algebra $\mathbf{A}$ is called a **value of $\mathbf{A}$ at** $a \in A$ if $\mathbf{B}$ is maximal with respect to not containing a. An easy lemma (Lemma 6.1 of [13]) asserts that $\mathbf{B}$ is a value of $\mathbf{A}$ if and only if B is completely meet-irreducible in the lattice of subalgebras of $\mathbf{A}$. Lemma 6.2 of [13] shows that every maximal element of a failset U in $\mathbf{D}^2$ is a value of $\mathbf{D}^2$.

We are now ready to identify the globally minimal failsets for $\mathbb{ISP}(\mathbf{4})$ relative to $\mathbb{S}(\mathbf{4}^2)$.

Proposition 4.1. *$\{m\}$ is a globally minimal failset (that is, m is absolutely unavoidable).*

Proof. First note that $m = m^{\smile}$. We shall appeal to Theorem 6.17 of [13] (or see [4], 8.5.12). This tells us that $\{m\}$ is a globally minimal failset provided that the following hold:

(i) m is not the joint kernel of two non-extendable partial endomorphisms,
(ii) $\mathbf{m}$ satisfies condition (H),
(iii) $\mathbf{m}$ is a value of $\mathbf{4}^2$, and
(iv) $\mathbb{S}(\mathbf{4}^2) \setminus \{m\}$ is closed under the action of e.

The only relations which are joint kernels are $\operatorname{graph}(e)$, $(\operatorname{graph}(e))^{\smile}$, the diagonal i and $j = \ker(e, e)$. None of these is m, so (i) is satisfied. For (iii) note that $\mathbf{m}$ is a value at $(0, 1)$.

Condition (ii) is most easily verified by Priestley duality. Suppose we have a homomorphism $x\colon \mathbf{m} \to \mathbf{4}$. Then $H(x)\colon H(\mathbf{4}) \to H(\mathbf{m})$ is an order-preserving map preserving the action of the g-map. Thus $H(x)$ is either a map onto a three-element chain in $H(\mathbf{m})$, or is a constant map onto a g-fixpoint. There are just two three-element chains in $H(\mathbf{m})$; the dual Kleene morphisms from $H(\mathbf{4})$ onto these two chains are $H(\rho_1)$ and $H(\rho_2)$. The constant map onto the g-fixpoint in $\operatorname{Im} H(\rho_i)$ is $H(e \circ \rho_i)$, for $i = 1, 2$. This accounts for all possible Kleene space morphisms from $H(\mathbf{4})$ to $H(\mathbf{m})$, so (H) holds.

Finally, suppose for a contradiction that $e \cdot s = m$ for some $\mathbf{s} \in \mathbb{S}(\mathbf{4}^2)$. Since $(b, 0) \in m$ we have $(1, 0) = (e(b), 0) \in s$. But $(1, 0) = (e(1), 0) \in s$ implies $(1, 0) \in m$, which is false. $\square$

In identifying other globally minimal failsets we employ the following lemma. Part (1) comes from The r-on-s Lemma from [13] while part (2) is Lemma 2.4(b) in [13] (see also [4], 8.1.2 and Exercise 8.9).

LEMMA 4.2. *Let* $\mathbf{D}$ *be a finite algebra and* $\underset{\sim}{\mathbf{D}}$ *an alter ego for* $\mathbf{D}$. *Let* $\mathbf{r}, \mathbf{s} \leqslant \mathbf{D}^2$.

(1) *Let* $x_1, x_2 \in D(\mathbf{s})$. *Then the following are equivalent:*
   (a) $(x_1, x_2) \in r^{D(\mathbf{s})}$;
   (b) *there is a (necessarily unique) homomorphism, namely* $\gamma\colon \mathbf{s} \to \mathbf{r}$, *given by* $\gamma(a) = (x_1(a), x_2(a))$ $(a \in \mathbf{s})$, *such that* $x_i = \rho_i \circ \gamma$ *for* $i = 1, 2$.

(2) *Let* $\mathbf{s}$ *be a retract of* $\mathbf{r}$, *with associated maps* $\gamma\colon \mathbf{s} \rightarrowtail \mathbf{r}$ *and* $\lambda\colon \mathbf{r} \twoheadrightarrow \mathbf{s}$ *with* $\lambda \circ \gamma = \mathrm{id}_{\mathbf{s}}$. *Then, for every map* $u\colon D(\mathbf{s}) \to M$,

$$\mathrm{Fail}_{\mathbf{r}}(v) = \mathrm{Fail}_{\mathbf{s}}(u) \text{ where } v := u \circ D(\gamma),$$

so that any failset of $\mathbf{s}$ *which contains* $\mathbf{r}$ *is also a failset of* $\mathbf{r}$.

The following lemma records very elementary facts about failsets within $\mathbb{S}(\mathbf{4}^2)$.

LEMMA 4.3. *Let* U *be a failset in* $\mathbb{S}(\mathbf{4}^2)$ *not containing the endomorphism* e. *Then neither the relation* i *nor the relation* j *belongs to* U *and no decomposable subalgebra of* $\mathbf{4}^2$ *belongs to* U.

Proof. In the following and in subsequent arguments of the same sort, we use the fact that $r \vdash s$ implies that any failset containing s also contains r.

We have $e \vdash j$ because $j = \ker(e, e)$. Also $e \vdash d$ because $e \cap e^{\smile} = d$. Then $e \vdash c$ because c is a trivial expansion of $\{0, 1\}$, which is entailed by d by repetition removal. Therefore c (and so also $c^{\smile}$) cannot belong to U. Also $\{0, 1\} \vdash \{0, 1\}^2$, so the latter is not in U. Finally, the trivial relations b and i are not in U. □

The Kleene spaces dual to $\mathbf{f}$, $\mathbf{g}$ and $\mathbf{k}$ are shown in Figure 4.

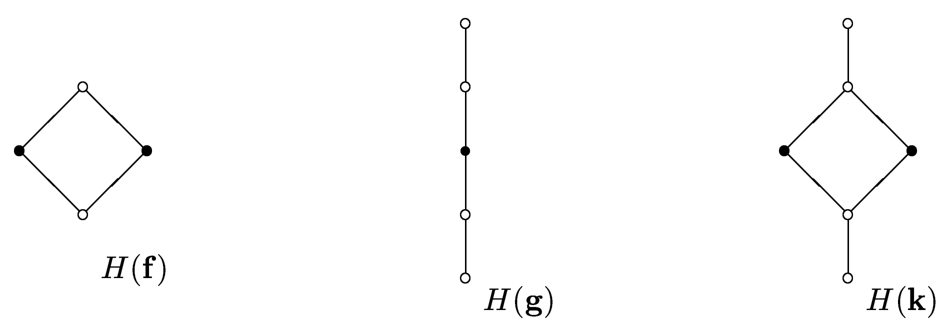


FIGURE 4

PROPOSITION 4.4. *Each of the sets*

$$U_1 := \{f, k, k^{\smile}, \ell, \ell^{\smile}\} \quad \text{and} \quad U_2 := \{g, g^{\smile}, k, k^{\smile}, \ell, \ell^{\smile}\}$$

is a globally minimal failset (within $\mathbb{S}(\mathbf{4}^2)$*). Furthermore, every globally minimal failset (within* $\mathbb{S}(\mathbf{4}^2)$*) not containing* e *and containing an element of either* U_1 *or* U_2 *coincides with* U_1 *or* U_2.

Proof. We must show that each of U_1 and U_2 is a minimal failset of each of its members ([13], Theorem 3.14 or [4], Theorem 8.3.8).

We see from Figure 3 that $\ell \cap \ell^{\smile} = f = k \cap k^{\smile}$. Therefore any failset containing f must contain U_1.

Also $j \cap \ell = g = j \cap k$. Since $e \vdash j$, any failset not containing e and containing g must contain U_2.

We claim that any failset of k or of ℓ must contain either f or g. To do this it suffices to show that $g \cdot f = k$ and $f \cdot g = \ell$ are homomorphic relational products. For the former of these we use the binary homomorphism $F\colon \mathbf{k} \to \mathbf{4}$ given by

$$\begin{array}{llll} F(0,0)=0, & F(a,0)=0, & & \\ & F(a,a)=a, & F(b,a)=b, & \\ & F(a,b)=a, & F(b,b)=b, & \\ & & F(b,1)=1 & F(1,1)=1. \end{array}$$

For the latter the appropriate binary homomorphism $G\colon \boldsymbol{\ell} \to \mathbf{4}$ is

$$\begin{array}{llll} G(0,0)=0, & G(a,0)=a, & G(b,0)=a, & \\ & G(a,a)=a, & G(b,a)=a, & \\ & G(a,b)=b, & G(b,b)=b, & \\ & G(a,1)=b, & G(b,1)=b, & G(1,1)=1. \end{array}$$

We conclude that U_1 and U_2 are minimal failsets of each of their members, provided each is a failset of every one of its members (rather than just a weak failset).

It is easily seen that each of $\mathbf{f}$ and $\mathbf{g}$ is a subretract of both $\mathbf{k}$ and $\boldsymbol{\ell}$. One method is to show dually that the required retraction maps exist; see Figures 2 and 4 for the Kleene spaces involved. By Lemma 4.2(2), any failset of f or of g is also a failset of k and of ℓ.

We claim that U_1 is a failset of f. The dual $D(\mathbf{f})$ consists of the four distinct elements $\rho_1, \rho_2, e \circ \rho_1, e \circ \rho_2$. We define $u\colon D(\mathbf{f}) \to 4$ by putting $u(\rho_1) = a$ and $u(x) = 0$ for $x \in D(\mathbf{f}) \setminus \{\rho_1\}$. Since $(\rho_1, \rho_2) \in f_{D(\mathbf{f})}$ and $(a, 0) \notin f$, the map u does not preserve f, and hence, from above, does not preserve k, ℓ or their converses. By construction, u preserves e, and so also any relation entailed by e. Also u preserves any relation which contains $\{0, a\}^2$, that is, any relation which has j as a subset. This leaves us to prove that both g and h are preserved by u.

Suppose for a contradiction that we had $(x_1, x_2) \in g^{D(\mathbf{f})}$ and $(u(x_1), u(x_2)) \notin g$. The latter condition forces $u(x_1) = 0$ and $u(x_2) = a$, so that $x_2 = \rho_1$ and $x_1 \neq \rho_1$. In order for $(x_1, x_2) \in g^{D(\mathbf{f})}$ to hold, there must exist a homomorphism $\gamma\colon \mathbf{f} \to \mathbf{g}$ with $\gamma(p, q) = (x_1(p, q), x_2(p, q))$ for all $(p, q) \in f$ (by Lemma 4.2(1)). Then $\gamma(p, q)$ is given, for all (p, q), by one of (q, p), $(e(p), p)$ or $(e(q), p)$. None of these possibilities is compatible with $\operatorname{Im}\gamma \subseteq \mathbf{g}$, so we have the required condition. Thus u preserves g. In a similar way, suppose for a contradiction that we had $(x_1, x_2) \in h^{D(\mathbf{f})}$ and $(u(x_1), u(x_2)) \notin h$. This time we conclude that $x_1 = \rho_1$. Assume there were a homomorphism $\gamma\colon \mathbf{f} \to \mathbf{h}$ with $\gamma(p, q) = (p, x_2(p, q))$ for all $(p, q) \in f$. It is routine to check that none of the four possibilities p, q, $e(p)$, $e(q)$ for $x_2(p, q)$ gives a map with image in $\mathbf{h}$. Thus we have a contradiction and deduce that u preserves h. We have proved that U_1 is $\mathrm{Fail}_{\mathbf{f}}(u)$.

Arguing in a similar way we can show that U_2 is a failset of $\mathbf{g}$. The dual $D(\mathbf{g})$ has three elements: ρ_1, ρ_2 and $e \circ \rho_1 = e \circ \rho_2$. We define $u\colon D(\mathbf{g}) \to 4$ by $u(\rho_2) = a$, $u(\rho_1) = u(e \circ \rho_1) = 0$. This preserves all the relations from $\mathbb{S}(\mathbf{4}^2)$ except those in U_2 and, maybe, f or h. We may use Lemma 4.2(1) in the same manner as in the preceding paragraph to show that u does preserve f and h. Consequently $U_2 = \mathrm{Fail}_{\mathbf{g}}(u)$.

The final statement follows from what has already been proved. □

PROPOSITION 4.5. *The only globally minimal failsets which do not contain e are those given in Propositions* 4.1 *and* 4.4.

Proof. Suppose that U is a globally minimal failset such that $e \notin U$ and $m \notin U$. We know from Proposition 4.4 that the only globally minimal failsets containing ℓ are those given in that proposition, so assume that $\ell \notin U$.

Since every failset must contain a value, or equivalently a meet-irreducible element of $\mathbb{S}(\mathbf{4}^2)$, we must have one of the relations c, j, n in U. However Lemma 4.3 implies that $c \notin U$ and $j \notin U$. An elementary computation shows that $e \cdot \ell^{\smile} = n$ and that this is a homomorphic relational product. Hence a failset containing neither e nor ℓ also cannot contain n. □

Since we are interested exclusively in which relations must be included in a dualising set in addition to the elements of the endomorphism monoid, we need not investigate failsets containing endomorphisms. Because of this, Theorem 4.6 is stated in a slightly weaker form than it could be: the endomorphism cannot be dropped, and the specified dualities are in fact optimal.

THEOREM 4.6. *Each of the following sets gives a dualising set for* $\mathbf{4}$ *for which the resulting duality for* $\mathfrak{L}$ *is such that none of the non-endomorphisms can be discarded from any of the sets without destroying the duality:*

$$\{e, m, f, g\}, \quad \{e, m, k\}, \quad \textit{and} \quad \{e, m, \ell\}.$$

5. ENDODUALISABLE AND FINITE ENDOPRIMAL KLEENE ALGEBRAS IN $\mathfrak{L}$

In this section we shall prove the following two theorems.

THEOREM 5.1 (ENDODUALISABILITY THEOREM FOR $\mathfrak{L}$). *Let* $\mathbf{M}$ *be a finite non-Boolean Kleene algebra in* $\mathfrak{L}$. *Then the following are equivalent:*

(a) $\mathbf{M}$ *is endodualisable;*
(b) $\mathbf{M}$ *has as retracts each of* $\mathbf{m}$, $\mathbf{f}$ (*the free algebra on one generator*) *and* $\mathbf{g}$ (*the six-element chain*).

THEOREM 5.2 (ENDOPRIMALITY THEOREM FOR $\mathfrak{L}$). *Let* $\mathbf{M}$ *be a finite non-Boolean Kleene algebra in* $\mathfrak{L}$. *Then the following are equivalent:*

(a) $\mathbf{M}$ *is endoprimal;*
(b) $\mathbf{M}$ *is* 2*-endoprimal;*
(c) $\mathbf{M}$ *has as retracts both* $\mathbf{f}$ (*the free algebra on one generator*) *and* $\mathbf{g}$ (*the six-element chain*).

Moreover, either of the following two conditions is sufficient for $\mathbf{M}$ *to be endoprimal:*

(d) $\mathbf{M}$ *has* $\mathbf{k}$ *as a retract;*
(e) $\mathbf{M}$ *has* $\boldsymbol{\ell}$ *as a retract.*

We shall first separate out some steps in the proofs as lemmas. With the aid of these results we shall also be able to give further information about the algebras which are endoprimal but not endodualisable.

LEMMA 5.3. *Let* $\mathbf{M}$ *be a finite Kleene algebra in* $\mathfrak{L}$. *Assume that* $\mathbf{M}$ *does not have the six-element chain* $\mathbf{6}$ *as a retract,* (*equivalently, the ideal* $\mathbf{M}^{\wedge}$ (*the filter* $\mathbf{M}^{\vee}$) *is a Boolean lattice*). *Then*

(1) $\mathbf{M}$ *is non-endodualisable, with the test algebra* $\mathbf{6}$ *witnessing this fact.*
(2) $\mathbf{M}$ *is not* 2*-endoprimal.*

Proof. We consider $\mathbf{D} = \mathbf{4}$ as a subalgebra of $\mathbf{M}$, with $\mathbf{M}^{\wedge} = [0, a]$. We take our test algebra to be $\mathbf{6}$, and regard this as a subalgebra of $\mathbf{M}^2$ by identifying it with its isomorphic copy

$$\mathbf{g}_D = \{(0,0),(a,0),(a,a),(b,b),(b,1),(1,1)\} \leqslant \mathbf{M}^2.$$

To simplify notation, we henceforth write g_D simply as g, and write the elements of $\mathbf{6}$ as $0 < c < d < \neg d < \neg c < 1$. Note that $\mathbf{6}^{\wedge} = [0, d]$ and, for any $x \in D(\mathbf{6}) = \hom(\mathbf{6}, \mathbf{M})$, we have $x(d) \in \mathbf{M}^{\wedge}$ since $x(d) \wedge \neg x(d) = x(d \wedge \neg d) = x(d)$.

We define a map $u\colon D(\mathbf{6}) \to M$ by taking, for $x \in D(\mathbf{6})$, the image $u(x)$ to be the relative complement of the element $x(c)$ in the interval $[0, x(d)]$ of the Boolean lattice $\mathbf{M}^{\wedge}$. To show that u preserves $\operatorname{End}\mathbf{M}$, let $e \in \operatorname{End}\mathbf{M}$ and $(x_1, x_2) \in (\operatorname{graph}(e))^{D(\mathbf{6})}$. Then $e(x_1(c)) = x_2(c)$ and $e(x_1(d)) = x_2(d)$. By definition, $u(x_2)$ is the relative complement of $x_2(c)$ in the interval $[0, x_2(d)]$, hence it is the relative complement of $e(x_1(c))$ in the interval $[e(0), e(x_1(d))] \subseteq \mathbf{M}^{\wedge}$. Since $e{\restriction}_{\mathbf{M}^{\wedge}}$ maps $\mathbf{M}^{\wedge}$ into itself, it obviously commutes with the relative complementation on $\mathbf{M}^{\wedge}$. Therefore we get that $u(x_2) = e(u(x_1))$, that is, $(u(x_1), u(x_2)) \in \operatorname{graph}(e)$, as required. We have $(\rho_1, \rho_2) \in g^{D(\mathbf{6})}$, but $(u(\rho_1), u(\rho_2)) = (0, a) \notin g$. Hence u does not preserve g, and we conclude that $\mathbf{M}$ is non-endodualisable. This completes the proof of (1).

Now consider (2). Using the description of $H(\mathbf{F}_{\mathfrak{K}}(2))$ (see Section 3 and Lemma 3.3(2)) we see easily that $\mathbf{F}_{\mathfrak{L}}(2) = \mathbf{F}_{\mathfrak{K}}(2)$ has $\mathbf{6}$ as a retract. By (1) and the Retraction Test Algebra Lemma (see 2.2(1)), $\mathbf{F}_{\mathfrak{L}}(2)$ is a test algebra witnessing non-endodualisability of $\mathbf{M}$. Thus $\mathbf{M}$ is not 2-endoprimal. □

LEMMA 5.4. *A finite non-Boolean Kleene algebra* $\mathbf{M}$ *is* 1*-endoprimal if and only if it has* $\mathbf{F}_{\mathfrak{K}}(1)$ *as a retract.*

Proof. If $\mathbf{M}$ has $\mathbf{F}_{\mathfrak{K}}(1)$ as a retract then $\operatorname{End}\mathbf{M}$ yields a duality on $\mathbf{F}_{\mathfrak{K}}(1)$, hence $\mathbf{M}$ is 1-endoprimal by Proposition 1.1. For the converse we distinguish two cases according to Lemma 3.3(3).

We first assume that $H(\mathbf{M})$ has a unique fixpoint α. Then $\mathbf{M}^{\wedge} = \{a \in M \mid \alpha \notin a\}$ and $\mathbf{M}^{\vee} = \{a \in M \mid \alpha \in a\}$, whence $\mathbf{M} = \mathbf{M}^{\wedge} \oplus \mathbf{M}^{\vee}$. To show that $\mathbf{M}$ is not 1-endoprimal, define $u\colon M \to M$ by

$$u(a) = \begin{cases} 0 & \text{if } a \in M^{\vee}, \\ 1 & \text{if } a \in M^{\wedge}. \end{cases}$$

Since any endomorphism maps $\mathbf{M}^{\vee}$ ($\mathbf{M}^{\wedge}$) into $\mathbf{M}^{\vee}$ ($\mathbf{M}^{\wedge}$), the map u preserves $\operatorname{End}\mathbf{M}$. Because $\mathbf{M}$ is not Boolean there exists $a \in \mathbf{M}$ such that $0 < a < \neg a < 1$ (take $a = H(\mathbf{M})^{\wedge}$). This means that u coincides with none of the unary term functions $x \mapsto 0$, $x \mapsto 1$, $x \mapsto x$, $x \mapsto \neg x$, $x \mapsto x \wedge \neg x$ and $x \mapsto x \vee \neg x$.

Now we assume that $H(\mathbf{M})$ has at least two g-fixpoints in $H(\mathbf{M})$, no two of which have a common upper (or lower) bound. We first show that there exist indecomposable algebras $\mathbf{M}_1, \dots, \mathbf{M}_q$ in $\mathbb{ISP}(\mathbf{4})$, not all of which are Boolean, such that, for each $i = 1, \dots, q$,

(i) $H(\mathbf{M}_i)$ has a unique fixpoint, thus $\mathbf{M}_i = {\mathbf{M}_i}^{\wedge} \oplus {\mathbf{M}_i}^{\vee}$;
(ii) $\mathbf{M}$ is (isomorphic to) a subalgebra of $\mathbf{M}_1 \times \cdots \times \mathbf{M}_q$;
(iii) $\mathbf{M}^{\wedge} = {\mathbf{M}_1}^{\wedge} \times \cdots \times {\mathbf{M}_q}^{\wedge}$.

Let the g-fixpoints in $Y = H(\mathbf{M})$ be $\alpha_1, \dots, \alpha_k$. For $i = 1, \dots, k$, let $Y_i = {\uparrow}\alpha_i \cup {\downarrow}\alpha_i$ and define $\mathbf{M}_i = K(Y_i)$. Then Y, as a set, is the disjoint union of $Y_1, \dots, Y_k$. Further, by transitivity of the order, for each i, j with $i \neq j$, no point of ${Y_i}^{\wedge}$ (${Y_i}^{\vee}$) is comparable to any point of ${Y_j}^{\wedge}$ (${Y_j}^{\vee}$). All of (i)–(iii) now follow. Some $\mathbf{M}_i$ is non-Boolean since $\mathbf{M}$ is not Boolean.

We now show that for each endomorphism e of $\mathbf{M}$ there exist a map $\varepsilon\colon \{1, \dots, q\} \to \{1, \dots, q\}$ and homomorphisms $e_i\colon \mathbf{M}_{\varepsilon(i)} \to \mathbf{M}_i$ such that

(E) $\qquad e((a_1, \dots, a_q)) = (e_1(a_{\varepsilon(1)}), \dots, e_q(a_{\varepsilon(q)})) \quad$ for all $(a_1, \dots, a_q) \in M$.

First note that $H(e)$ maps each g-fixpoint α_i to some g-fixpoint, $\alpha_{\varepsilon(i)}$ say. If $y \in Y_i$, then $y \geqslant \alpha_i$ or $y \leqslant \alpha_i$. Assume without loss of generality that $y \geqslant \alpha_i$. Then $H(e)(y) \geqslant \alpha_{\varepsilon(i)}$. We conclude that $H(e)$ maps Y_i into $Y_{\varepsilon(i)}$, and so its restriction to Y_i is the dual of a Kleene homomorphism $e_i\colon M_{\varepsilon(i)} \to M_i$. This is exactly what we need to obtain the desired formula for e above.

Now, for each $i = 1, \dots, q$, define $v_i\colon \mathbf{M}_i \to \mathbf{M}_i$ by

$$v_i(a_i) = \begin{cases} \neg a_i & \text{if } a_i \in {\mathbf{M}_i}^{\vee}, \\ 0 & \text{if } a_i \in {\mathbf{M}_i}^{\wedge}; \end{cases}$$

note that this defines v_i completely, by (i) above. We can now define $u\colon \mathbf{M} \to \mathbf{M}$ by $u := (v_1 \sqcap \cdots \sqcap v_q){\restriction}_{\mathbf{M}}$. This is well defined by (iii) above.

We claim that u is not a term function. Suppose, without loss of generality, that $\mathbf{M}_1$ is non-Boolean and let $0 < a < \neg a < 1$ in $\mathbf{M}_1$. Then, in particular,

$$u((a, 0, \dots, 0)) = (0, \dots, 0),$$
$$u((\neg a, 1, \dots, 1)) = (a, 0, \dots, 0).$$

So, certainly u cannot be a term function: every unary term function f satisfies either $f = f \circ \neg$ or $f = \neg \circ f \circ \neg$.

We claim that u preserves $\operatorname{End}\mathbf{M}$. Let $e \in \operatorname{End}\mathbf{M}$. Thus e is given by formula (E) above. Fix $(a_1, \dots, a_q) \in \mathbf{M}$. Fix i and let $j = \varepsilon(i)$. Then

$$\begin{aligned} \pi_i(e(u(a_1, \dots, a_q))) &= e_i(v_j(a_j)) \\ &= \begin{cases} 0 & \text{if } a_j \in {\mathbf{M}_j}^{\wedge}, \\ e_i(\neg a_j) & \text{if } a_j \in {\mathbf{M}_j}^{\vee} \end{cases} \\ &= \begin{cases} 0 & \text{if } a_j \in {\mathbf{M}_j}^{\wedge}, \\ \neg e_i(a_j) & \text{if } a_j \in {\mathbf{M}_j}^{\vee}. \end{cases} \end{aligned}$$

On the other hand, noting that e_i maps $\mathbf{M}_j{}^{\wedge}$ $(\mathbf{M}_j{}^{\vee})$ into $\mathbf{M}_i{}^{\wedge}$ $(\mathbf{M}_i{}^{\vee})$, we have

$$\begin{aligned}\pi_i(u(e(a_1,\dots,a_q))) &= v_i(e_i(a_j))\\ &= \begin{cases} 0 & \text{if } a_j \in \mathbf{M}_j{}^{\wedge},\\ \neg e_i(a_j) & \text{if } a_j \in \mathbf{M}_j{}^{\vee}.\end{cases}\end{aligned}$$

We deduce that $\mathbf{M}$ is not 1-endoprimal. □

Lemma 5.5. *Let* $\mathbf{M}$ *be a finite algebra in* $\mathbb{ISP}(\mathbf{4})$ *and let* $Y = H(\mathbf{M})$. *Then the following are equivalent:*

(a) $\mathbf{M}$ *is indecomposable* (*equivalently,* Y *is order-connected*) *and* $\mathbf{M}$ *does not have* $\mathbf{m}$ *as a retract;*
(b) $Z := Y^{\wedge} \cup Y^{\circ} = \{\, y \in Y \mid y \geqslant g(y) \,\}$ *is order-connected.*

Proof. We may assume that $\mathbf{M}$ is non-Boolean, since the only Boolean algebra satisfying the conditions of either (a) or (b) has $|\mathbf{M}| \leqslant 2$, when both conditions hold trivially.

We first prove that (a) implies (b). Since $\mathbf{M}$ is indecomposable, Y is order-connected and $|Y| > 1$ since $\mathbf{M}$ is not Boolean. Every point of Y is comparable to a g-fixpoint (by Lemma 3.1), and every g-fixpoint is non-isolated and so is neither maximal nor minimal.

Consider the ordered set $Z := \{x \in Y \mid x \geqslant g(x)\}$, and let its order components be $Z_1, \dots, Z_p$. Let $Y_i := Z_i \cup g(Z_i)$. Then $Y_1, \dots, Y_p$ is a partition of Y into disjoint sets. Suppose for a contradiction that $p > 1$. Since Y is connected, some point u of Y_1 is comparable to some point v of, say, Y_2. We may assume that $u > v$ and that $u > g(u)$, $u \in Z_1$, and $v < g(v)$, $v \in g(Z_2)$. We may assume also that, with respect to the order, u is minimal and v maximal among pairs u, v satisfying the specified conditions. This implies that u covers v and $g(v)$ covers $g(u)$ in the order. Since $\mathbf{M}$ lies in $\mathbb{ISP}(\mathbf{4})$, there are g-fixpoints w and z such that $g(u) < w < u$ and $v < z < g(v)$. Then $\{u, v, g(u), g(v), w, z\}$ forms a subset of Y (not necessarily convex) which is order-isomorphic to $H(\mathbf{m})$. Let $Y_1' := Y_2 \cup \dots \cup Y_p$. We can retract Y_1 onto the three-element chain $g(u) < w < u$ and can retract Y_1' onto the three-element chain $v < z < g(v)$. Putting together these retractions we get a retraction of Y onto $H(\mathbf{m})$, contrary to hypothesis. Consequently $p = 1$, as required.

Conversely, assume that Z is order-connected. Certainly this implies that Y is also order-connected. Now assume for a contradiction that Y has a subset W isomorphic to $H(\mathbf{m})$, onto which it retracts by means of a Kleene space morphism φ. Let the two g-fixpoints in W be ν and $\nu^{\smile}$. Then the connectedness of Z and the fact that every point of Z majorises a g-fixpoint together imply that there are g-fixpoints $z_1, \dots, z_n$ such that $z_1 = \nu$, $z_n = \nu^{\smile}$, and z_i and z_{i+1} have a common upper bound (necessarily in Z), for $i = 1, \dots, n-1$. For each i, the images $\varphi(z_i)$ and $\varphi(z_{i+1})$ must be g-fixpoints in W with a common upper bound, and hence must coincide. We deduce that φ cannot map onto W, contrary to hypothesis, so that $\mathbf{m}$ is not a retract of $\mathbf{M}$. □

Lemma 5.6. *Suppose* $\mathbf{M}$ *is a finite non-Boolean Kleene algebra in* $\mathbb{ISP}(\mathbf{4})$ *which does not contain* $\mathbf{m}$ *as a retract. Then* $\mathbf{M}$ *is not endodualisable.*

Proof. We first assume that $\mathbf{M}$ is indecomposable. We shall use $\mathbf{m}$ as test algebra. We regard $\mathbf{m}$ as a subalgebra of $\mathbf{M}^2$, by retracting $\mathbf{M}$ onto $\mathbf{4} \leqslant \mathbf{M}$, and identifying $\mathbf{m}$ with $\mathbf{4}^2 \setminus \{(0,1),(1,0)\}$. We investigate $D(\mathbf{m}) = \hom(\mathbf{m},\mathbf{M})$ by considering the Kleene space maps $H(x)$ where $x \in \hom(\mathbf{m},\mathbf{M})$. By Lemma 5.5, $Y := H(\mathbf{M})$ is the union of the sets $Z = \{x \in Y \mid x \geqslant g(x)\}$ and $g(Z) = \{x \in Y \mid x \leqslant g(x)\}$, where Z and $g(Z)$ are order-connected. Now let $\varphi\colon H(\mathbf{M}) \to H(\mathbf{m})$ be a Kleene space map. Any two points in Z are linked by a fence in Z, and likewise for $g(Z)$. Since φ maps fences to fences, maps g-fixpoints to g-fixpoints, and preserves g, we see that either $\operatorname{Im}\varphi \subseteq \{\lambda,\nu,\mu\}$ or $\operatorname{Im}\varphi \subseteq \{\lambda^{\smile},\nu^{\smile},\mu^{\smile}\}$, the labelling being as in Figure 2. Thus the elements f of $\hom(\mathbf{m},\mathbf{M})$ are of two mutually exclusive types: those for which $\operatorname{Im} H(f) \subseteq \{\lambda,\nu,\mu\}$ (type I) and those for which $\operatorname{Im} H(f) \subseteq \{\lambda^{\smile},\nu^{\smile},\mu^{\smile}\}$ (type II). The projection maps ρ_1,ρ_2 from $\mathbf{m} \leqslant \mathbf{M}^2$ are of different types; without loss of generality ρ_1 is of type I. Now define $u\colon D(\mathbf{m}) \to M$ by letting $u(f) = 0$ if f is of type I and $u(f) = 1$ if f is of type II. Then u preserves $\operatorname{End}\mathbf{M}$, because f and $e \circ f$ are of the same type for any endomorphism e of $\mathbf{M}$, as is clear by looking at the dual map $H(f) \circ H(e)$. However u does not preserve $\mathbf{m}$: $(\rho_1,\rho_2) \in m^{D(\mathbf{m})}$ but $(u(\rho_1),u(\rho_2)) = (0,1) \notin m$. Thus $\mathbf{m}$ is not entailed by $\operatorname{End}\mathbf{M}$.

Now assume that $\mathbf{M}$ is decomposable and write $\mathbf{M}$ as the product of indecomposable algebras $\mathbf{M}_1,\dots,\mathbf{M}_q$ in $\boldsymbol{\mathcal{L}}$. Write Y_i for $H(\mathbf{M}_i)$ and pick a g-fixpoint $\alpha_i \in Y_i$ $(i = 1,\dots,q)$. Form a new Kleene space Z from $Y = H(\mathbf{M})$ by identifying the points $\alpha_1,\dots,\alpha_q$ to form a single point, α say. Let $\mathbf{N} = K(Z)$ be the associated Kleene algebra. The space Z is order-connected, so $\mathbf{N}$ is indecomposable, by Lemma 3.4. By construction, $\mathbf{N}$ is also non-Boolean since some $\mathbf{M}_i$ is. Further, if we could retract $\mathbf{N}$ onto $\mathbf{m}$ then we could retract some $\mathbf{M}_i$ onto $\mathbf{m}$ too. To see this, note that if Z were retractable onto $H(\mathbf{m})$ then the co-retraction would have to map $H(\mathbf{m})$ into a single $H(\mathbf{M}_i)$. Also, if $\mathbf{M}_i$ were to retract onto $\mathbf{m}$ then $\mathbf{M}$ would too, since we can retract $\mathbf{M}$ onto $\mathbf{M}_i$ (the dual of a suitable retraction map can be defined by sending all points of $Y \setminus Y_i$ onto α_i). Hence $\mathbf{N}$ does not have $\mathbf{m}$ as a retract. By the first part of the proof applied to $\mathbf{N}$, we deduce that $\mathbf{N}$ is not endodualisable. As any subalgebra of an indecomposable algebra in $\boldsymbol{\mathcal{L}}$ is indecomposable (use Lemma 3.4), and the other condition in Theorem 2.2(3) is satisfied too, by Lemma 3.5, we obtain that $\mathbf{N}^q$ is not endodualisable.

Now note that $\mathbf{M}$ is a retract of $\mathbf{N}^q$. Thus if $\mathbf{M}$ were endodualisable, $\mathbf{N}^q$ would be too (see Theorem 2.2(2)). Since this is not so, we conclude that $\mathbf{M}$ is not endodualisable. □

We now have all the ingredients we need for the proofs of Theorems 5.1 and 5.2.

Proof of Theorem 5.1. The implication (b) ⇒ (a) is immediate from Theorem 4.6 and Theorem 2.3. Now assume that (b) fails. If $\mathbf{M}$ fails to have $\mathbf{m}$ as a retract then $\mathbf{M}$ is not endodualisable by Lemma 5.6. If $\mathbf{M}$ fails to have $\mathbf{f} = \mathbf{F}_{\mathcal{K}}(1)$ as a retract then it is not 1-endoprimal, by Lemma 5.4, and hence not endodualisable, by Proposition 1.1. Likewise, if $\mathbf{M}$ fails to have $\mathbf{g} = \mathbf{6}$ as a retract then it is not 2-endoprimal, by Lemma 5.3 and hence not endodualisable. □

Proof of Theorem 5.2. The implication (a) ⇒ (b) is trivial. Suppose (c) fails. If $\mathbf{M}$ fails to have $\mathbf{g} = \mathbf{6}$ as a retract then by Lemma 5.3 $\mathbf{M}$ is not 2-endoprimal. If $\mathbf{M}$

fails to have $\mathbf{f} = \mathbf{F}_{\mathcal{K}}(1)$ as a retract then by Lemma 5.4, $\mathbf{M}$ is not 1-endoprimal and hence certainly not 2-endoprimal. Hence (b) implies (c).

Finally we use Proposition 2.6 to prove that each of (c), (d) and (e) implies (a). By Theorem 4.6 and Theorem 2.3, $\mathbf{M}$ is dualised by End M together with the set of relations S_D where $\mathbf{D} = \mathbf{4}$ and S is any one of the sets $\{m, f, g\}$, $\{m, k\}$ or $\{m, \ell\}$. The following easily derived formulae show that m may be obtained from f, g, from k or from ℓ by taking relational products:

$$m = (f \cdot g) \cdot (f \cdot g)^{\smile},$$
$$m = (k \cdot k^{\smile}) \cdot k,$$
$$m = (\ell \cdot \ell^{\smile}).$$

Therefore Proposition 2.6 applies. (Alternatively, we can show that (d) implies (c) and (e) implies (c) by noticing that each of $\mathbf{f}$ and $\mathbf{g}$ is a retract of both $\mathbf{k}$ and $\boldsymbol{\ell}$). □

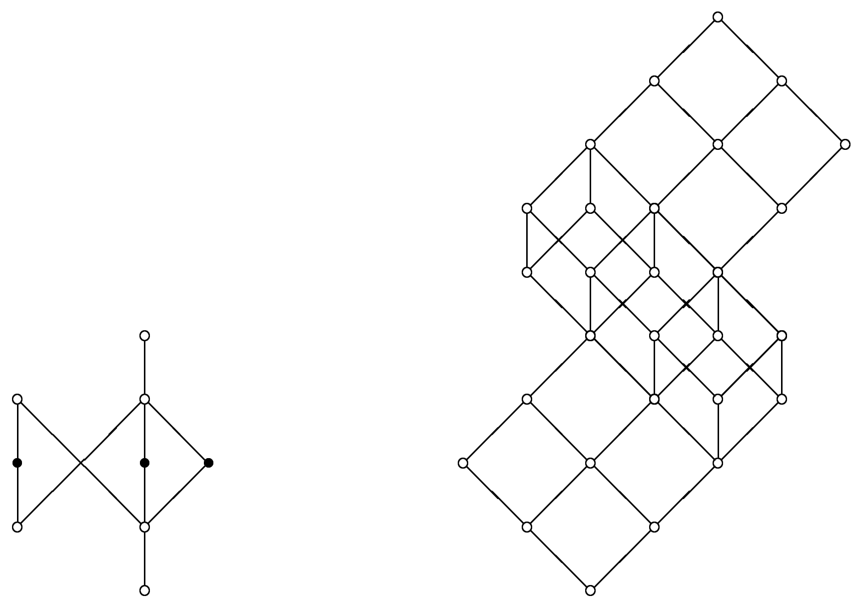


FIGURE 5

Clearly it is easy to use Theorem 5.1 to construct non-Boolean endodualisable Kleene algebras. For example, $\mathbf{m} \times \mathbf{6} \times \mathbf{F}_{\mathcal{K}}(1)$ is such an algebra, with 504 elements. Figure 5 shows (the lattice reduct of) a smaller one, with 34 elements. This algebra was obtained by first looking for a Kleene space with $H(\mathbf{m})$, $H(\mathbf{f})$ and $H(\mathbf{g})$ as retracts; this dual is shown alongside.

We remark that it is easy to prove that none of the free algebras $\mathbf{F}_{\mathcal{K}}(k)$ $(k \geqslant 1)$ is endodualisable. It is also easy to find small algebras in $\mathcal{L}$ which are endoprimal but not endodualisable: both $\mathbf{k}$ and $\boldsymbol{\ell}$ are examples of such algebras. So too are $\mathbf{6} \times \mathbf{F}_{\mathcal{K}}(1)$ and the indecomposable algebra $\mathbf{p}$ having dual shown in Figure 6; each of these certainly retracts onto each of $\mathbf{f}$ and $\mathbf{g}$, but not onto $\mathbf{m}$.

Since each of $\mathbf{k}$, $\boldsymbol{\ell}$ and $\mathbf{p}$ retracts onto both $\mathbf{f}$ and $\mathbf{g}$, it is clear that each of the following conditions

(i) $\mathbf{M}$ has $\mathbf{m}$ and $\mathbf{k}$ as retracts

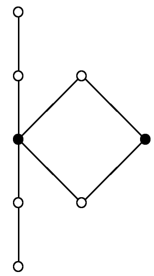


FIGURE 6

(ii) $\mathbf{M}$ has $\mathbf{m}$ and $\boldsymbol{\ell}$ as retracts
(iii) $\mathbf{M}$ has $\mathbf{m}$ and $\mathbf{p}$ as retracts

is sufficient for $\mathbf{M} \in \mathcal{L}$ to be endodualisable. It is natural to ask whether these conditions are also necessary. Obviously, the dual of the algebra in Figure 5 shows that (ii) and (iii) are not necessary. The given endodualisable algebra has $\mathbf{m}$ and $\mathbf{k}$ as retracts but has neither $\boldsymbol{\ell}$ nor $\mathbf{p}$ as a retract. The algebra $\mathbf{m} \times \boldsymbol{\ell}$ whose Priestley dual is a disjoint union of the Priestley duals of $\mathbf{m}$ and $\boldsymbol{\ell}$ depicted in Figure 2 shows that the condition (i) above is not necessary for $\mathbf{M} \in \mathcal{L}$ to be endodualisable, too.

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