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Endoprimal distributive lattices are endodualisable

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Abstract. L. Márki and R. Pöschel have characterised the endoprimal distributive lattices as those which are not relatively complemented. The theory of natural dualities implies that any finite algebra A on which the endomorphisms of $\mathbf{A}$ yield a duality on the quasivariety $\mathbb{ISP}(A)$ is necessarily endoprimal. This note investigates endodualisability for finite distributive lattices, and shows, in a manner which elucidates Márki and Pöschel's proof, that it is equivalent to endoprimality.

Let $(A; F)$ be an algebra and denote its endomorphism monoid by End A. Then A is said to be **endoprimal** if every finitary function on A commuting with each element of End A is a term function, and k-**endoprimal** ($k \geq 1$) if every function of arity not greater than k commuting with all endomorphisms is a term function. In [15], L. Márki and R. Pöschel proved that the following statements are equivalent for a non-trivial distributive lattice $(L; \vee, \wedge)$:

(1) L is endoprimal, and
(2) L is not relatively complemented.

The contrapositive of the implication (1) $\Rightarrow$ (2) is easy (by exploiting relative complementation, as a ternary function). As Márki and Pöschel point out, their result shows that an endoprimal algebra need not generate a quasivariety with the kind of nice structural properties that are obtained when primality is extended in other ways, notably to quasiprimality (see [16]). Thus the occurrence and role of endoprimal algebras seems a little mysterious, and examples are rather scarce. However there is one situation in which endoprimal algebras arise very naturally, through duality theory. When we refer to a duality, we shall always mean a natural duality in the sense of Davey and Werner. For our purposes, the recent survey [4] is the most convenient reference. A finite algebra A is said to be **endodualisable** if End A yields a duality on $\mathscr{A} = \mathbb{ISP}(A)$, in the sense defined in [4], Section 2. If this

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is the case, then, for each $n \geq 1$, the free algebra $F\mathscr{A}(n)$ is exactly the set of endomorphism-preserving maps from A^n to A ([4], Lemma 2.3). Hence any endodualisable algebra A is endoprimal. Endodualisability is a rather rare phemonemon (see [9], [11]). Nevertheless it can occur: a finite subdirectly irreducible Heyting algebra is endodualisable if and only if it is either a chain or $2^2 \oplus 1$ ([3], [4], pp. 104–105, [10]). It is therefore natural to ask whether the concepts of endoprimality and endodualisability coincide for finite distributive lattices. We shall let $\mathbf{D}$ denote the variety of $\{0, 1\}$-distributive lattices, and $\mathbf{D}'$ be the variety of all distributive lattices. Parallel endoprimality and endodualisability results hold for the two varieties. To expose the connections with [15], we shall concentrate on $\mathbf{D}'$. At the end of the paper we consider $\mathbf{D}$ (and some other examples too).

We prove the following theorem.

THEOREM 1. *Let* $(L; \vee, \wedge)$ *be a finite non-trivial distributive lattice. Then the following are equivalent*:

(1) L *is* 3-*endoprimal*;

(2) L *is endoprimal*;

(3) L *is endodualisable*;

(4) *the retractions of* L *onto* $\{0, 1\}$ *together with the constants* 0, 1 *yield a duality on* $\mathbf{D}' = \mathbb{ISP}(L)$;

(5) L *is not a boolean lattice.*

Our proof uses the cycle of implications

$$(1) \Rightarrow (5) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1).$$

Trivially, (4) $\Rightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). To obtain the contrapositive of (1) $\Rightarrow$ (5) we may follow [15] and argue as follows. Let $f(a, b, c)$ be the complement of a in $[a \wedge b \wedge c, a \vee b \vee c]$. Then $f : L^3 \to L$ cannot be a term function, since it does not preserve the underlying order, but it does commute with all endomorphisms. Thus to complete the cycle we only have to prove that (5) $\Rightarrow$ (4). Our strategy for this is to show that a suitable subset of the endomorphisms of L, together with the order relation, $s := \{(0, 0), (0, 1), (1, 1)\}$, on $\{0, 1\} \subseteq L$ yield a duality on $\mathbf{D}' = \mathbb{ISP}(L)$. This then reduces the problem to that of showing that, when L is non-boolean, the relation s is redundant. As we show below (see Proposition 4, Corollary 5 and Remark 6), the proof that this is so relies on essentially the same arguments, but lifted up to a duality-theoretic setting, as are employed by Márki and Pöschel in part of their proof.

In order to set up a natural duality for quasivariety $\mathscr{A} := \mathbb{ISP}(A)$, where $(A; F)$ is a finite algebra, we proceed as follows. We seek a set R of algebraic relations on

A (that is, a subset R of $\bigcup_{n\geq 1}\mathbb{S}(A^n)$) such that every algebra in $\mathscr{A}$ is isomorphic, via the natural evaluation map, to an algebra of continuous R-preserving maps. In general, the problem of describing a suitable dualising set R, or proving that no such set exists, is very difficult. However in case A has a reduct in a variety $\mathscr{D}$ for which a duality is already available, the piggyback methodology of Davey and Werner can frequently be utilized to set up a duality for $\mathbb{ISP}(A)$ (see [13], [14], or [4]). The proof of Theorem 2 is self-contained, but implicitly uses the piggyback strategy.

We shall regard endomorphisms either as maps or as algebraic binary relations, as expedient, and shall replace constant endomorphisms by their images, regarded as distinguished elements.

We define Γ to be the set of retractions of L onto $\{0, 1\}$. We let $s = \{(0, 0), (0, 1), (1, 1)\}$ be the order on $\{0, 1\} \subseteq L$; it may be regarded either as a subalgebra of L^2 or as a subalgebra of 2^2.

THEOREM 2. *Let* $(L; \wedge, \vee) \in \mathbf{D}'$ *with* $|L| > 1$. *Then* $\Gamma \cup \{0, 1\} \cup \{s\}$ *(regarded as a set of algebraic relations on* $\mathbf{L}$*) yields a duality on* $\mathbf{D}' = \mathbb{ISP}(L)$.

Proof. Let $A \in \mathbf{D}'$ and let $u : D(A) \to L$ preserve $\Gamma \cup \{0, 1\} \cup \{s\}$, where $D(A) := \mathbf{D}'(A, L)$. Since $\{0, 1\} \subseteq L$, the set $H(A) := \mathbf{D}(A, 2)$ can be identified with a subset of $D(A)$. If $x \in H(A)$, then for each $g \in \Gamma$ we have $x = g \circ x$, whence

$$u(x) = u(g \circ x) = u(g(x)) = g(u(x)) \in \{0, 1\}.$$

Thus $u\restriction_{H(A)} : H(A) \to \{0, 1\}$ preserves s, 0 and 1. Since, by the Priestley duality for $\mathbf{D}'$ (see [12], pp. 216–217), the set $\{s\} \cup \{0, 1\}$ yields a duality on $\mathbf{D}'$, it follows that there exists $a \in A$ such that $u(x) = x(a)$ for all $x \in H(A)$. Since Γ separates the points of L, in order to show that $u(x) = x(a)$ for all $x \in D(A)$ it suffices to show that $g(u(x)) = g(x(a))$ for all $g \in \Gamma$. Let $x \in D(A)$ and let $g \in \Gamma$. Then

$$\begin{aligned} g(u(x)) &= u(g(x)) && \text{as } u \text{ preserves } g \\ &= u(g \circ x) && \text{by definition of } g(x) \\ &= (g \circ x)(a) && \text{as } g \circ x \in H(A) \\ &= g(x(a)), \end{aligned}$$

as required. □

The duality for $\mathbf{D}'$ based upon L rather than 2 may be viewed as a modified form of Priestley duality. The "Priestley" part of this modified duality is the relation s which, as the order relation on 2, is a remnant of the original Priestley duality for $\mathbf{D}'$ based upon 2. Viewed in this way, Theorems 1 and 2 say that we may remove "Priestley" from (the modified) Priestley duality precisely when the lattice L upon which the duality is based is nonboolean. We shall see in Proposition 7 that although "Priestley" cannot be removed in the boolean case, she can at least be replaced by a finitary algebraic relation which, like $\underset{\sim}{s}$, is nonboolean.

Certainly L is endodualisable whenever the retractions, Γ, together with the constant endomorphisms onto 0 and 1, suffice to yield a duality. The key to investigating when this is so is the test algebra technique introduced in [8] and further developed in [9]. This exploits the fact that any algebraic relation t on an algebra A may be viewed as an algebra, $\underset{\sim}{t}$, in $\mathscr{A} := \mathbb{ISP}(A)$. Given a set R of algebraic relations on A, the dual $D(\underset{\sim}{t})$ of the algebra $\underset{\sim}{t}$ is the set of homomorphisms $\mathscr{A}(\underset{\sim}{t}, A)$, on which the relations in R are defined to act pointwise. We recall (see [9], Section 2) that, if $\mathscr{B} := \bigcup_{n \geq 1} \mathbb{S}(A^n)$, $R \subseteq \mathscr{B}$, and $t \in \mathscr{B}$, then, by definition, R **entails** t (written $R \vdash t$), if, for every algebra $C \in \mathscr{A}$, every continuous map $u : D(C) \to A$ which preserves R also preserves t, and R **entails** t **on** $D(\underset{\sim}{t})$ if every map $u : D(\underset{\sim}{t}) \to A$ which preserves R also preserves t.

LEMMA 3 (**The Test Algebra Lemma**, [9], Lemma 2.3, Lemma 4.1). *Let $\mathscr{A} = \mathbb{ISP}(A)$, where A is a finite algebra. Let $R \subseteq \mathscr{B} := \bigcup_{n \geq 1} \mathbb{S}(A^n)$ and let $t \in \mathscr{B}$. Then R entails t if and only if R entails t on $D(\underset{\sim}{t})$. In particular, if $R \cup \{t\}$ yields a duality on $\mathscr{A}$ and R entails t on $D(\underset{\sim}{t})$, then R yields a duality on $\mathscr{A}$.*

Let L be a finite distributive lattice. For any n-ary algebraic relation t on L we let $\rho_1^t, \dots, \rho_n^t$ be the members of $D(\underset{\sim}{t})$ obtained by restricting to t the projections $\pi_1, \dots, \pi_n : L^n \to L$. As before we denote by s the subalgebra $\{(0,0), (0,1), (1,1)\}$ of L^2. Note that the relation s on $D(\underset{\sim}{s})$ is simply the partial order whose associated strict order is $0 < \rho_1^s < \rho_2^s < 1$. The following result serves two purposes. It allows us to complete the proof of Theorem 1, by invoking Corollary 5 (of which a more direct proof shall also be given). It also yields information which we need in our investigation below of alternative dualising sets for $\mathbf{D}'$. The constants 0 and 1 have been included in the statement of this theorem only to simplify the proof. With a little extra work (see remark 6) the constants can be omitted.

PROPOSITION 4. *Let L be a non-trivial finite distributive lattice and let r be an algebraic relation on L with $|r| > 1$ such that $\underset{\sim}{r}$ is non-boolean. Then $\Gamma \cup \{0, 1\} \cup \{r\}$ entails s.*

Proof. Let r be an n-ary relation as described in the statement of the proposition. Let $u : D(\underset{\sim}{s}) \to L$ preserve $\Gamma \cup \{0, 1\} \cup \{r\}$. If $x \in \mathbf{D}(\underset{\sim}{s}, L)$ with $\operatorname{im} x \subseteq \{0, 1\}$, then for each $g \in \Gamma$ we have $x = g \circ x$, whence $u(x) \in \{0, 1\}$ (as in the proof of

Theorem 2). Hence $u(\rho_1^s)$ and $u(\rho_2^s)$ belong to $\{0, 1\}$. Suppose by way of contradiction that u does not preserve s. Since the relation s on $D(\underline{s})$ is the order satisfying $0 < \rho_1^s < \rho_2^s < 1$ and since $u(0) = 0$ and $u(1) = 1$, we must have $u(\rho_1^s) = 1$ and $u(\rho_2^s) = 0$. Since $\underline{r}$ is non-boolean, there exist prime filters F, G of $\underline{r}$ with $F \subsetneq G$. Since $\underline{r}$ is a sublattice of L^n (in $\mathbf{D}'$), both F and G extend to prime filters of L^n (see [7], Exercise 10.6(ii)(a) or [1], p. 74). Hence (see [7], Exercise 9.3) there exists i and a prime filter F_i of L such that

$$F = r \cap (L \times \cdots L \times F_i \times L \cdots \times L)$$

and similarly there exists j and a prime filter G_j of L such that

$$G = r \cap (L \times \cdots L \times G_i \times L \cdots \times L).$$

Let $\gamma : \underline{s} \to \underline{r}$ be a homomorphism satisfying $\gamma((0, 0)) \notin G$, $\gamma((0, 1)) \in G \backslash F$ and $\gamma((1, 1)) \in F$. Then $\rho_i^r \circ \gamma$, $\rho_j^r \circ \gamma \in D(\underline{s})$ with $\chi_{F_i}(\rho_i^r \circ \gamma) = \rho_1^s$ and $\chi_{G_i}(\rho_j^r \circ \gamma) = \rho_2^s$. Thus

$$1 = u(\rho_1^s) = \chi_{F_i}(u(\rho_i^r \circ \gamma)) \quad \text{and} \quad 0 = u(\rho_2^s) = \chi_{G_j}(u(\rho_j^r \circ \gamma)).$$

Since $(\rho_1^r \circ \gamma, \ldots, \rho_n^r \circ \gamma) \in r$ on $D(\underline{s})$ and u preserves r, we finally get

$$(u(\rho_1^r \circ \gamma), \ldots, u(\rho_n^r \circ \gamma)) \in r \cap (F \backslash G),$$

a contradiction. □

COROLLARY 5. *If* $(L; \vee, \wedge)$ *is a finite distributive lattice which is non-boolean, then* $\Gamma \cup \{0, 1\}$ entails s.

Proof. Consider the special case of Proposition 4 in which $r = L$. If $u : D(\underline{s}) \to L$ preserves $\Gamma \cup \{0, 1\}$ then u preserves $\Gamma \cup \{0, 1\} \cup \{L\}$, whence u preserves s since L is non-boolean. □

REMARK 6. A more direct proof of the fact that *if* L *is non-boolean, then* Γ *alone* (*without the constants* 0 *and* 1) *entails the relation* s may be obtained as follows. Denote the relation s on $D(\underline{s})$ simply by $\leq$. As in the proof of Proposition 4, let F and G be prime filters of L with $F \subsetneq G$ and let $c \in G \backslash F$. It is easy to check that if $v_1, v_2 \in \{0, \rho_1, \rho_2, 1\}$ with $v_1 \leq v_2$, then there exists $v_3 \in D(\underline{s})$ such that $\operatorname{im} v_3 \subseteq \{0, c, 1\}$ and $\chi_F \circ v_3 = v_1$ while $\chi_G \circ v_3 = v_2$. For example: for $0 < \rho_1$, we choose $v_3(0, 0) = v_3(0, 1) = 0$ and $v_3(1, 1) = c$; for $\rho_1 < 1$ we choose $v_3(0, 0) =$

$v_3(0, 1) = c$ and $v_3(1, 1) = 1$; for $0 < 1$ we choose the constant map $v_3(0, 0) = v_3(0, 1) = v_3(1, 1) = c$; for $\rho_1 < \rho_2$ we choose $v_3(0, 0) = 0$, $v_3(0, 1) = c$ and $v_3(1, 1) = 1$. (The map v_3 in the final example corresponds to the map γ in the proof of Proposition 4.) Now let $u : D(\underset{\sim}{s}) \to L$ preserve χ_F and χ_G, and let $v_1 \leq v_2$. Then there exists $v_3 \in D(\underset{\sim}{s})$ and $\chi_F(v_3) = v_1$ and $\chi_G(v_3) = v_2$. Thus

$$u(v_1) = u(\chi_F(v_3)) = \chi_F(u(v_3)) \leq \chi_G(u(v_3)) = u(\chi_G(v_3)) = u(v_2),$$

whence u preserves s. Thus $\{\chi_F, \chi_G\}$, and therefore Γ, entails s on $D(\underset{\sim}{s})$, as required. The ideas underpinning this proof are further developed in Davey, Haviar and Priestley [6].

We may pursue further the discussion of dualising sets for $\mathbf{D}'$. Fix a finite lattice L in $\mathbf{D}'$. We have shown in Theorem 2 that, by regarding $\mathbf{D}'$ as $\mathbb{ISP}(L)$, we may obtain a dualising set drawn from subalgebras of L^2 by taking the set Γ of retractions onto $\{0, 1\}$, the constants 0, 1, and the relation s, the last being needed only when L is boolean. The results of [9] allow us to characterise those algebraic relations r on L which suffice, with the endomorphisms, to yield a duality on $\mathbf{D}'$.

PROPOSITION 7. *Let* $(L; \vee, \wedge)$ *be a finite boolean lattice. Let* $\underset{\sim}{r}$ *be a subalgebra of* L^n *from some* $n \geq 1$. *Then* $\Gamma \cup \{0, 1\} \cup \{r\}$ *yields a duality on* $\mathbf{D}'$ *if and only if* r *satisfies one (and hence all) of the following equivalent conditions:*

(1) $\underset{\sim}{r}$ *is not a boolean lattice;*

(2) $\underset{\sim}{r}$ *is not closed under (coordinatewise) relative complementation in* L^n;

(3) $\underset{\sim}{r}$ *retracts onto a 3-element chain.*

Proof. The equivalence of (1)–(3) is elementary. To prove the duality claim, define a map $u : \mathbf{D}'(\underset{\sim}{s}, L) \to L$ by declaring $u(x)$ to be the relative complement of $x((0, 1))$ in the interval $[x((0, 0)), x((1, 1))]$, for all $x \in \mathbf{D}'(\underset{\sim}{s}, L)$. Then, in the notation of [9], the failset

$$U := \mathrm{Fail}_{\underset{\sim}{s}}(u) = \{r \in \mathscr{R} \mid u \text{ fails to preserve } r\}$$

consists exactly of the sublattices not closed under relative complementation (which include s). By Lemma 2.4 of [9], for each $r \in \mathrm{Fail}_{\underset{\sim}{s}}(u)$ there exists $v : D(\underset{\sim}{r}) \to L$, namely $v := u \circ D(\gamma)$, where $\gamma \in \mathbf{D}'(\underset{\sim}{s}, \underset{\sim}{r})$, such that $\mathrm{Fail}_{\underset{\sim}{r}}(v) = \mathrm{Fail}_{\underset{\sim}{s}}(u)$. Thus U is a failset of each of its members. We claim that U is a minimal failset of each $r \in U$. Let

$$r \in \mathrm{Fail}_{\underset{\sim}{r}}(v) \subseteq U.$$

If $g \in \Gamma$, then $\underset{\sim}{g} \cong L$ and hence $g \notin U$. Thus v preserves g. If v preserves some finitary non-boolean algebraic relation r', then (by Proposition 4) v preserves s and hence is an evaluation map (since $\Gamma \cup \{0, 1\} \cup \{s\}$ yields a duality on $\mathbf{D}'$). But then v preserves r, a contradiction. Hence every finitary non-boolean algebraic relation is in $\mathrm{Fail}_{\underline{r}}(v)$. That is, $U \subseteq \mathrm{Fail}_{\underline{r}}(v)$, as required.

We deduce, from the Optimal Duality Theorem of [9], that any duality for $\mathbb{ISP}(L)$ using relations in $\mathscr{B}$ must contain at least one relation from U. Because a 1-element lattice is boolean, U contains no 1-element sets, and so in particular U contains neither 0 nor 1. Since U also contains no endomorphisms of L and we know that $\Gamma \cup \{0, 1\} \cup \{s\}$ does yield a duality, Theorem 4.4 of [9] implies that $\Gamma \cup \{0, 1\} \cup \{r\}$ does so for any $r \in U$. □

Our arguments above have yielded the following corollary, which is worth noting explicitly.

COROLLARY 8. *Let L be a finite non-trivial boolean lattice and let R be a family of finitary algebraic relations on L. If R yields a duality on $\mathbf{D}'$, then R includes a non-boolean relation.*

We now consider the modifications that need to be made to obtain parallel results for $\mathbf{D}$. The main theorem is now as follows.

THEOREM 9. *Let $(L; \vee, \wedge, 0, 1)$ be a finite distributive lattice. Then the following are equivalent:*

(1) *L is 1-endoprimal;*
(2) *L is endoprimal;*
(3) *L is endodualisable;*
(4) *the retractions of L onto $\{0, 1\}$ yield a duality on $\mathbf{D} = \mathbb{ISP}(L)$;*
(5) *L is not a boolean lattice.*

This theorem is proved by the same cycle of implications as is Theorem 1. To obtain the contrapositive of (5) ⇒ (1), we use the unary function given by complementation. In place of Theorem 2 we need the analogous theorem that the retractions together with s yield a duality for $\mathbf{D}$; it is proved in the same manner. Finally, observe that there is an analogue of Corollary 5 for $\mathbf{D}$, proved in just the same way. Some further gloss on this result is informative. The algebra $\underline{s}$ is a 3-element chain. Under the obvious bijection, θ, from the dual $D(\underline{s}) := \mathbf{D}(\underline{s}, L)$ to L the pointwise action of endomorphisms on $D(\underline{s})$ corresponds to the functional action of endomorphisms on L. Further, for $x_1, x_2 \in D(\underline{s})$,

$$\begin{aligned}(x_1, x_2) \in s \text{ on } D(\underline{s}) \quad &\Leftrightarrow \quad (\forall a \in \underline{s})(x_1(a), x_2(a)) \in s \\ &\Leftrightarrow \quad (\theta(x_1), \theta(x_2)) \in s.\end{aligned}$$

Hence it is clear that $\Gamma \vdash s$ on $D(\underset{\sim}{s})$ if and only if every map $u : L \to L$ commuting with all $g \in \Gamma$ (and so necessarily mapping into $\{0, 1\}$) is order-preserving. It is worth noting that 1-endoprimality of L ensures that this is true. Thus 1-endoprimality implies endodualisability – the implication (1) $\Rightarrow$ (4) in Theorem 9.

Proposition 7 is also valid, *mutatis mutandis*, with $\mathbf{D}$ in place of $\mathbf{D}'$: in condition (2), substitute 'complement' for 'relative complement'. Observe that, once $D(\underset{\sim}{s})$ is identified with L, we recognise the map u as being complementation on L. According to the proposition we may augment the endomorphisms of a boolean lattice L with any non-boolean algebraic relation on L to obtain a duality for $\mathbf{D}$. Our initial choice of supplementary relation – the order $\underset{\sim}{s}$ on $\{0, 1\} \subseteq L$ – is a natural one. We remark that another natural choice is the order relation, t say, on L itself. For ordered sets P_1, P_2, we denoted by $\langle P_1, P_2 \rangle$ the set of order-preserving maps from P_1 to P_2 with the pointwise order. Recall that, for finite distributive lattices $L_i \cong \langle P_i, 2 \rangle$ $(i = 1, 2)$, the coproduct $L_1 * L_2$ in $\mathbf{D}$ may be described either as $\langle P_1, L_2 \rangle$ or as $\langle P_2, L_1 \rangle$ (see [2]). Observe that $\underset{\sim}{t} \cong \langle 2, L \rangle$. We deduce that $\underset{\sim}{t}$ is the coproduct of $\underset{\sim}{s}$ and L. The dual space $D(\underset{\sim}{t})$ is then in bijective correspondence with $L \times \operatorname{End} L$, and endomorphisms and other algebraic relations are easily seen to act on this in the expected way.

It is a corollary of Theorems 1 and 9 that a finite distributive lattice is $\mathbf{D}$-endoprimal if and only if it is $\mathbf{D}'$-endoprimal. It is tempting to try to prove this directly, thereby making it unnecessary to give parallel proofs of the two theorems. The following argument partially achieves this, but is not entirely satisfactory since it involves the non-boolean property, albeit in an elementary way. It uses ideas drawn from [15]; the same ideas can also be used to give an alternative proof of Proposition 7.

PROPOSITION 10. *A finite distributive lattice* $(L; \vee, \wedge)$ *is endoprimal* (*in* $\mathbf{D}'$) *if and only if* $(L; \vee, \wedge, 0, 1)$ *is endoprimal* (*in* $\mathbf{D}$).

Proof. Let $(L; \vee, \wedge)$ be endoprimal. We show that $(L; \vee, \wedge, 0, 1)$ is endoprimal. Let $f : L^n \to L$ preserve all $\{0, 1\}$-endomorphisms on L. If f is a constant map equal to 0 or 1, then f clearly is a term function of $(L; \vee, \wedge, 0, 1)$. Assume now that this is not the case. We will show that if $e \in \operatorname{End}(L; \vee, \wedge)$ then $e(f(x_1, \ldots, x_n)) = f(e(x_1), \ldots, e(x_n))$, that is, that $g(e(f(x_1, \ldots, x_n))) = g(f(e(x_1), \ldots, e(x_n)))$ for every $g \in \Gamma$, which will obviously complete the proof. There are two cases to consider.

If $g \circ e$ is a $\{0, 1\}$-endomorphism then we have

$$g(e(f(x_1, \ldots, x_n))) = f(g \circ e(x_1), \ldots, g \circ e(x_n)) = g(f(e(x_1), \ldots, e(x_n)))$$

since f preserves $g \circ e$ and g.

Now let $g \circ e$ be a constant map of L onto $\{0\}$ or $\{1\}$, for example let $g \circ e \equiv 0$. Then

$$f(g \circ e(x_1), \ldots, g \circ e(x_n)) = f(0, \ldots, 0).$$

It suffices to show that $f(0, \ldots, 0) = 0$. Since $(L; \vee, \wedge)$ is endoprimal, L is non-boolean (otherwise the unary non-term function of complementation would preserve $\mathrm{End}(L; \vee, \wedge)$, a contradiction). Hence there exists a pair F, G of prime filters of L with $F \subsetneq G$. The characteristic functions χ_F, χ_G of F, G belong to Γ, thus f preserves χ_F, χ_G. This yields, since f is a total map, that f also preserves

$$\chi_F^{\smile} \circ \chi_G = \{(\chi_F(c), \chi_G(c)) \mid c \in L\} = \{(0, 0), (0, 1), (1, 1)\} = s,$$

the order relation on $\{0, 1\}$. Clearly, $f(0, \ldots, 0) \in \{0, 1\}$. Suppose that $f(0, \ldots, 0) = 1$. Then $f \restriction_{\{0,1\}^n} \equiv 1$ since f preserves the order s on $\{0, 1\}$. But then for any $g \in \Gamma$ and any $x_1, \ldots, x_n \in L^n$ we get

$$g(f(x_1, \ldots, x_n)) = f(g(x_1), \ldots, g(x_n)) = 1,$$

whence f is a constant function equal to 1, a contradiction.

The converse is trivial. □

We end with some simple observations on the non-distributive case.

PROPOSITION 11. *The pentagon N_5 and the diamond M_3 are not 1-endoprimal and therefore not endodualisable.*

Proof. Let $L = \{0, a, b, c, 1\}$ with $0 < a < b < 1$ be a pentagon. The crucial fact is that non-trivial endomorphisms of L do not separate the elements a, b. Hence the endomorphisms are preserved by the function $f : L \to L$ defined by $f(a) = b$, $f(b) = a$ and $f(x) = x$ for $x \notin \{a, b\}$. This function clearly is not a lattice term, and therefore L is not 1-endoprimal. We conclude that L is not endodualisable.

For the diamond M_3 we in the same way use the function $f : M_3 \to M_3$ defined by $f(0) = 1$, $f(1) = 0$ and $f(x) = x$ if $x \notin \{0, 1\}$. □

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