

# WHEN IS A FULL DUALITY STRONG?

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ABSTRACT. The relationship between full and strong dualities in the theory of natural dualities is not yet understood. Our aim in this paper is to present partial solutions to the Full versus Strong Problem, which asks if every full duality is necessarily strong. We introduce local versions of this problem and prove that they have affirmative solutions for four well-known classes of algebras: abelian groups, semilattices, relative Stone Heyting algebras and bounded distributive lattices. Along the way we provide some useful additions to the general theory.

## 1. INTRODUCTION

A full (natural) duality is a special kind of dual category equivalence between a quasi-variety  $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\underline{\mathbf{M}})$  of algebras generated by a finite algebra  $\underline{\mathbf{M}}$  and a category  $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\underline{\mathbf{M}})$  of structured topological spaces generated by an *alter ego*  $\underline{\mathbf{M}}$  of  $\underline{\mathbf{M}}$ . Here  $\mathbb{I}$ ,  $\mathbb{S}$ ,  $\mathbb{P}$  (arbitrary index sets) and  $\mathbb{P}^+$  (non-empty index sets), denote the usual class operators while  $\mathbb{S}_c$  denotes (*topologically*) *closed substructure of*. This dual category equivalence is set up by a natural pair of contravariant hom-functors  $\mathbb{D} : \mathcal{A} \rightarrow \mathcal{X}$  and  $\mathbb{E} : \mathcal{X} \rightarrow \mathcal{A}$ . In general, the (discrete) topological structure  $\underline{\mathbf{M}} = \langle M; G, H, R, \mathcal{J} \rangle$  may have sets  $G$  of operations,  $H$  of partial operations and  $R$  of relations in its type. The structure  $\underline{\mathbf{M}}$  is required to be *algebraic over  $\underline{\mathbf{M}}$*  meaning that the operations in  $G$  and  $H$  are assumed to be homomorphisms with respect to the algebra  $\underline{\mathbf{M}}$  and the relations in  $R$  are assumed to be subalgebras of the appropriate powers of  $\underline{\mathbf{M}}$ . This makes all category theory work properly and a dual adjunction called a *pre-duality* is automatically set up between the categories  $\mathcal{A}$  and  $\mathcal{X}$  via the functors  $\mathbb{D}$  and  $\mathbb{E}$ . We refer to Chapter 1 of the text by Clark and Davey [3] for details.

If an alter ego  $\underline{\mathbf{M}}$  of  $\underline{\mathbf{M}}$  is found such that the evaluation map  $e_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbb{E}\mathbb{D}(\mathbf{A})$  is an isomorphism, for each algebra  $\mathbf{A}$  in  $\mathcal{A}$ , we say that  $\underline{\mathbf{M}}$  *yields a natural*

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2000 *Mathematics Subject Classification*. Primary: 08C05; Secondary: 08C15, 06D50.

*Key words and phrases*. Natural duality, full duality, strong duality, abelian group, semilattice, bounded distributive lattice, Heyting algebra.

The authors wish to thank Jane Pitkethly for her helpful comments. The second author gratefully acknowledges the support of a 2004 Institute for Advanced Study Fellowship at La Trobe University.

*duality on  $\mathcal{A}$*  based on  $\underline{\mathbf{M}}$  or that  $\underline{\mathbf{M}}$  *dualises*  $\underline{\mathbf{M}}$ . Dualisability of  $\underline{\mathbf{M}}$  provides a uniform way of representing each algebra  $\mathbf{A}$  in the quasi-variety  $\mathcal{A}$  as an algebra of continuous structure-preserving maps from the  $\mathcal{X}$ -structure  $\mathbf{D}(\mathbf{A})$  into  $\underline{\mathbf{M}}$ . If, in addition, the evaluation map  $\varepsilon_{\mathbf{X}} : \mathbf{X} \rightarrow \mathbf{DE}(\mathbf{X})$  is an isomorphism, for each topological structure  $\mathbf{X}$  in  $\mathcal{X}$ , then we say that  $\underline{\mathbf{M}}$  *yields a full duality* on  $\mathcal{A}$  based on  $\underline{\mathbf{M}}$  or that  $\underline{\mathbf{M}}$  *fully dualises*  $\underline{\mathbf{M}}$ . A full duality gives us a uniform way of representing each structure  $\mathbf{X}$  in the topological quasi-variety  $\mathcal{X}$  as a structure of (algebraic) homomorphisms from the  $\mathcal{A}$ -algebra  $\mathbf{E}(\mathbf{X})$  into  $\underline{\mathbf{M}}$ . Full dualities are not yet fully understood and the stronger notion of strong duality is often used instead. A *strong duality* is a full duality where in addition  $\underline{\mathbf{M}}$  is injective in  $\mathcal{X}$ . If a finite algebra  $\underline{\mathbf{M}}$  has an alter ego  $\underline{\mathbf{M}}$  that yields a (full/strong) duality on  $\mathcal{A}$ , we say that  $\underline{\mathbf{M}}$  is (*fully/strongly dualisable*).

Strong dualities have proven to be a powerful tool for the study of the quasi-varieties  $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$  and were introduced and firstly studied by Clark and Davey [2]. So far every single example of a full duality has turned out to be strong. So there is a natural question going back to the foundations of the theory of natural dualities.

1. **Question.** *Is every full duality strong?*

This question is known as the *Full versus Strong Problem* and it is one of the most tantalizing open questions in the theory of natural dualities. In its early history, the problem was formulated in several ways which turned out to be equivalent. The first formulation of the problem appeared in the seminal work on the natural dualities by Davey and Werner [14].

**Question 1'.** *If  $\underline{\mathbf{M}}$  yields a full duality between the categories  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$  and  $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underline{\mathbf{M}})$ , does it follow that  $\underline{\mathbf{M}}$  is injective in  $\mathcal{X}$ ?*

Clark and Krauss [4] introduced the notions of term-closed subsets of  $M^S$  and hom-closed subsets of  $M^S$  and proved that they are the same. Their formulation of the problem looks completely different.

**Question 1''.** *If  $\underline{\mathbf{M}}$  yields a full duality between the categories  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$  and  $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underline{\mathbf{M}})$ , does it follow that every closed substructure of a non-zero power of  $\underline{\mathbf{M}}$  is term-closed (hom-closed)?*

The equivalence of the Davey-Werner and Clark-Krauss formulations of Problem 1 was proved much later by Clark and Davey [2] who proved that, given a duality between  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$  and  $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underline{\mathbf{M}})$ , every closed substructure of a non-zero power of  $\underline{\mathbf{M}}$  is term-closed (hom-closed) if and only if  $\underline{\mathbf{M}}$  yields a full duality on  $\mathcal{A}$  and  $\underline{\mathbf{M}}$  is injective in  $\mathcal{X}$ . Now we simply say, whenever all these equivalent conditions hold, that  $\underline{\mathbf{M}}$  *yields a strong duality on  $\mathcal{A}$* . We again refer to the text by Clark and Davey [3] for the definitions of term-closed and hom-closed subsets

of  $M^S$  and for a detailed discussion of full and strong natural dualities. We also refer to Davey, Haviar and Willard [11, 12] for some recent developments on the relationship between full and strong dualities.

Our aim in this paper is to understand further the relationship between full and strong dualities. Since the problem in its global version has remained open for the past 25 years, we introduce local versions that might prove more tractable and fruitful.

**2. Question.** *For an arbitrary finite algebra  $\underline{\mathbf{M}}$  in your favourite class  $\mathcal{C}$  of algebras, is every full duality based on  $\underline{\mathbf{M}}$  necessarily strong?*

In Davey, Haviar and Willard [11] an example is given which shows that at the finite level it is possible to find a full but not strong duality. Before reformulating Question 2 at the finite level, we need to make precise just what “at the finite level” means. Let  $\underline{\mathbf{M}}$  be an alter ego of a finite algebra  $\underline{\mathbf{M}}$ . We say that  $\underline{\mathbf{M}}$  yields a duality at the finite level on  $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\underline{\mathbf{M}})$  based on  $\underline{\mathbf{M}}$  if the evaluation map  $e_{\mathbf{A}} : \mathbf{A} \rightarrow \text{ED}(\mathbf{A})$  is an isomorphism, for every finite algebra  $\mathbf{A}$  in  $\mathcal{A}$ . If  $\underline{\mathbf{M}}$  yields a duality at the finite level and, in addition,  $\varepsilon_{\mathbf{X}} : \mathbf{X} \rightarrow \text{DE}(\mathbf{X})$  is an isomorphism, for every finite structure  $\mathbf{X}$  in  $\mathcal{X} := \mathbb{I}\mathbb{S}_{\mathcal{C}}\mathbb{P}^+ \underline{\mathbf{M}}$ , then we say that  $\underline{\mathbf{M}}$  yields a full duality at the finite level on  $\mathcal{A}$  based on  $\underline{\mathbf{M}}$ . If  $\underline{\mathbf{M}}$  yields a full duality at the finite level, then the categories  $\mathcal{A}_{\text{fin}}$  and  $\mathcal{X}_{\text{fin}}$  are dually equivalent via the functors  $\text{D}$  and  $\text{E}$ . Since there are four possible definitions of strong at the finite level, a little care is required. As it turns out, all four are equivalent. The proof of the following theorem is a simple modification of the proof of the Closure Theorem [3, 3.1.3].

**Finite Level Closure Theorem.** *Let  $\underline{\mathbf{M}}$  be a finite algebra, let  $n \in \mathbb{N}$  and let  $X \subseteq M^n$ . Then the following are equivalent:*

- (1)  $X$  is term closed;
- (2)  $X$  is hom-closed;
- (3)  $X$  is closed under all finitary algebraic partial operations on  $\underline{\mathbf{M}}$ ;
- (4)  $X$  is closed under all  $|X|$ -ary algebraic partial operations on  $\underline{\mathbf{M}}$ .

We say that  $\underline{\mathbf{M}}$  yields a strong duality at the finite level if  $\underline{\mathbf{M}}$  yields a full duality at the finite level and  $\underline{\mathbf{M}}$  is injective in  $\mathcal{X}_{\text{fin}}$ . The First Strong Duality Theorem [3, 3.2.4] tells us that  $\underline{\mathbf{M}}$  yields a strong duality at the finite level if and only if  $\underline{\mathbf{M}}$  yields a duality at the finite level and the equivalent conditions of the Finite Level Closure Theorem hold for each substructure  $\mathbf{X}$  of a finite power of  $\underline{\mathbf{M}}$ . We can now pose the finite-level version of Question 2.

**3. Question.** *For an arbitrary finite algebra  $\underline{\mathbf{M}}$  in your favourite class  $\mathcal{C}$  of algebras, is every duality based on  $\underline{\mathbf{M}}$  that is full at the finite level necessarily strong at the finite level?*

The first solutions to these local versions of the Full versus Strong Problem were given for full dualities based on the three-element chain in the variety of bounded distributive lattices in Davey, Haviar and Willard [11]. The answer was shown to be affirmative to Question 2 and negative to Question 3. Here we provide affirmative answers to Questions 2 and 3 for full dualities based on an arbitrary finite algebra in three varieties of algebras: abelian groups, semilattices (with or without bounds) and relative Stone Heyting algebras. We also develop some general conditions under which full implies strong that have the potential to add to the list of solutions in the near future. Finally, we answer Question 2 in the affirmative for full dualities based on an arbitrary finite lattice in the variety of bounded distributive lattices. We remark that in the companion paper [8] by Davey, Haviar, Niven and Perkal, Question 3 will be shown to have a negative answer in the variety of bounded distributive lattices by producing full but not strong dualities at the finite level based on an arbitrary finite non-Boolean lattice.

There is a further, weaker version of Question 2 that deserves to be recorded here.

**4. Question.** *In your favourite class  $\mathcal{C}$  of algebras, is every fully dualisable finite algebra necessarily strongly dualisable?*

There are various classes in which it has been proved that every dualisable algebra is strongly dualisable; for example, the class consisting of all finite algebras that generate a congruence-distributive variety [2], the variety of commutative rings [5], and the classes of graph algebras and flat graph algebras [18]. In any such class the answer to Question 4 is trivially ‘yes’. But the answer has also been proved to be ‘yes’ in the class consisting of three-element unary algebras [19], a class in which not every duality can be upgraded to a strong duality. It should be noted that the finite-level variant of this question makes no sense since every finite algebra  $\underline{\mathbf{M}}$  is strongly dualised at the finite level by the alter ego  $\underline{\mathbf{M}} = \langle M; H, \mathcal{J} \rangle$ , where  $H$  consists of all finitary algebraic partial operation on  $\underline{\mathbf{M}}$ . (To prove this, just combine the Brute Force Duality Theorem [3, 2.3.1] and the Finite Level Closure Theorem, and use the fact that a partial operation entails its domain [3, page 59]).

## 2. SUFFICIENT CONDITIONS FOR FULL TO IMPLY STRONG

In this section we develop several conditions under which full implies strong. One such condition has been known since the Full versus Strong Problem was first stated back in 1980. (See Lemma 3.2.10 and Exercise 6.5 in [3]).

**2.1. Lemma.** *Let  $\underline{\mathbf{M}}$  be a finite algebra that is injective in the quasi-variety it generates. If  $\underline{\mathbf{M}}$  is an alter ego of  $\underline{\mathbf{M}}$  that fully dualises  $\underline{\mathbf{M}}$  [at the finite level], then  $\underline{\mathbf{M}}$  strongly dualises  $\underline{\mathbf{M}}$  [at the finite level].*

Within the theory of natural dualities, the concept of *structural entailment* has only recently been identified and intensively developed: see Davey, Haviar and Willard [11, 12]. Its importance has become evident with the revived interest, within the last five years, in the study of full and strong dualities—see, in chronological order, Willard [23], Davey and Haviar [7], Hyndman and Willard [17], Clark, Idziak, Sabourin, Szabó and Willard [5], Lampe, McNulty and Willard [18], Hyndman [15], Pitkethly [19], Quackenbush and Szabó [22], Davey, Haviar and Willard [11, 12], Hyndman [16], Davey, Haviar, Niven and Perkal [8] and Pitkethly and Davey [20]. While entailment, first analyzed in Davey, Haviar and Priestley [10], is an important tool with respect to *dualisability*, most applications of structural entailment arise in connection with *strong dualisability* and *full dualisability*. We recall here only a few definitions and results on structural entailment needed later on. For a full account of structural entailment we refer to [12].

Let  $\underline{\mathbf{M}}$  be a finite algebra and let  $\underline{\mathfrak{M}} = \langle M; G, H, R, \mathcal{T} \rangle$  be an alter ego of  $\underline{\mathbf{M}}$ . Let  $\mathbf{X}$  be a closed substructure of a non-zero power of  $\underline{\mathfrak{M}}$  and let  $s$  be an  $I$ -ary algebraic relation on  $\underline{\mathbf{M}}$  for some set  $I$ . We say that  $\underline{\mathfrak{M}}$  *entails*  $s$  on  $\mathbf{X}$  if each morphism  $\alpha : \mathbf{X} \rightarrow \underline{\mathfrak{M}}$  preserves  $s$ . Then  $\underline{\mathfrak{M}}$  *entails*  $s$  if it entails  $s$  on every structure of the form  $\mathbf{D}(\mathbf{A})$ , for  $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$ . Finally,  $\underline{\mathfrak{M}}$  *structurally entails*  $s$  if it entails  $s$  on every closed substructure of each non-zero power of  $\underline{\mathfrak{M}}$ . Thus structural entailment implies entailment. The converse fails as is shown by the following example taken from [11] and [12]. Let  $\underline{\mathbf{M}}$  be the three-element chain regarded as a bounded distributive lattice and let  $\underline{\mathfrak{M}} := \langle M; f, g, \mathcal{T} \rangle$ , where  $f$  and  $g$  are the non-identity endomorphisms of  $\underline{\mathbf{M}}$ . It is well known that  $\underline{\mathfrak{M}}$  entails the order relation  $\leq_{\{0,1\}}$  (see [9], [6], [3, 9.5.1]), but  $\underline{\mathfrak{M}}$  does not structurally entail  $\leq_{\{0,1\}}$ ; indeed,  $X = \{0, 1\}$  is a substructure of  $\underline{\mathfrak{M}}$  and the map  $\alpha : X \rightarrow \{0, 1\} \subseteq M$  given by the Boolean complementation is a morphism from  $\mathbf{X}$  to  $\underline{\mathfrak{M}}$  that does not preserve  $\leq_{\{0,1\}}$ .

Let  $A$  be a non-empty set and let  $\mathbf{X}$  be a closed substructure of  $\underline{\mathfrak{M}}^A$ . For all  $a \in A$ , we define  $\rho_a := \pi_a \upharpoonright_X$ , where  $\pi_a : M^I \rightarrow M$  is the natural projection. (To avoid confusion between different restrictions, we occasionally write  $\rho_a^{\mathbf{X}}$  instead of  $\rho_a$ .) The closed substructure  $\mathbf{X}$  of  $\underline{\mathfrak{M}}^A$  is said to be *hom-minimal* if the only morphisms from  $\mathbf{X}$  to  $\underline{\mathfrak{M}}$  are the maps  $\rho_a$ , for  $a \in A$ . If, in addition,  $\rho_a \neq \rho_b$ , for all  $a \neq b$  in  $A$ , (that is,  $X$  has no repetition of coordinates), then  $\mathbf{X}$  is called *balanced*. Hom-minimal and balanced subalgebras of powers of  $\underline{\mathbf{M}}$  are defined analogously. An  $n$ -ary algebraic relation on  $\underline{\mathbf{M}}$  is called hom-minimal or balanced if the corresponding subalgebra of  $\underline{\mathfrak{M}}^n$  is. The following lemma from [12] will prove useful.

**2.2. Lemma.** [12, 3.7] *Let  $\underline{\mathbf{M}}$  be a finite algebra and let  $\underline{\mathfrak{M}}$  be an alter ego of  $\underline{\mathbf{M}}$ . Let  $\mathbf{A} \in \mathbb{ISP}(\underline{\mathbf{M}})$  and let  $\mathbf{X}$  be a balanced substructure of  $\mathbf{D}(\mathbf{A})$ . If the map  $\varepsilon_{\mathbf{X}} : \mathbf{X} \rightarrow \mathbf{DE}(\mathbf{X})$  is an isomorphism, then  $\mathbf{X} = \mathbf{D}(\mathbf{A})$ .*

In Davey, Haviar and Willard [12] a detailed characterization of structural entailment is given. The result below gives just one of the seven equivalent conditions.

**2.3. Theorem.** [12, 2.4] *Consider an alter ego  $\underline{\mathbf{M}} := \langle M; G, H, R, \mathcal{T} \rangle$  of a finite algebra  $\underline{\mathbf{M}}$ . Let  $s$  be an  $n$ -ary algebraic relation on  $\underline{\mathbf{M}}$ , for some  $n > 0$ . Then  $\underline{\mathbf{M}}$  structurally entails  $s$  if and only if*

$$s = \{ (c_1, \dots, c_n) \in M^n \mid \underline{\mathbf{M}} \models \Phi(c_1, \dots, c_n) \}$$

for some conjunct of atomic formulae,  $\Phi(v_1, \dots, v_n)$ , in the language of  $\underline{\mathbf{M}}$ .

The *enriched partial clone* of a structure  $\underline{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$  is defined to be the enriched partial clone  $[G \cup H]$  generated by  $G \cup H$ . (It is discussed extensively on pages 21, 64 and 278–283 of Clark and Davey [3].) We require the following result from [11, 12].

**2.4. Theorem.** [12, 3.9] *Let  $\underline{\mathbf{M}}$  be a finite algebra and let  $\underline{\mathbf{M}}$  be an alter ego of  $\underline{\mathbf{M}}$ . Assume that  $\varepsilon_{\mathbf{X}} : \mathbf{X} \rightarrow \text{DE}(\mathbf{X})$  is an isomorphism for all finite structures  $\mathbf{X}$  in  $\mathcal{X}$ .*

- (i) *Let  $n \in \mathbb{N}$  and let  $s$  be an  $n$ -ary algebraic relation on  $\underline{\mathbf{M}}$ . If  $\underline{\mathbf{M}}$  structurally entails  $s$ , then every  $n$ -ary algebraic partial operation on  $\underline{\mathbf{M}}$  with domain  $s$  has an extension in the enriched partial clone of  $\underline{\mathbf{M}}$ .*
- (ii) *Every algebraic total operation on  $\underline{\mathbf{M}}$  is in the enriched partial clone of  $\underline{\mathbf{M}}$ .*

Throughout the remainder of this section let  $\underline{\mathbf{M}}$  be a finite algebra in the quasi-variety  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{D}})$  generated by a finite algebra  $\underline{\mathbf{D}}$  and assume that  $\underline{\mathbf{D}}$  is a subalgebra of  $\underline{\mathbf{M}}$ . It follows that  $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$ . Let  $\mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}}) = \{\omega_1, \dots, \omega_n\}$ . We shall simultaneously view each  $\omega_i$  as a homomorphism from  $\underline{\mathbf{M}}$  into  $\underline{\mathbf{D}}$  and as an endomorphism of  $\underline{\mathbf{M}}$ . Define

$$\omega := \omega_1 \sqcap \dots \sqcap \omega_n : \underline{\mathbf{M}} \rightarrow \underline{\mathbf{D}}^n \leq \underline{\mathbf{M}}^n$$

by  $\omega(a) := (\omega_1(a), \dots, \omega_n(a))$ , for all  $a \in M$ . Clearly, the homomorphism  $\omega$  is an embedding as the maps  $\omega_1, \dots, \omega_n$  separate the points of  $M$ . Let

$$M_r := \omega(M) \subseteq D^n \subseteq M^n$$

and let  $\sigma : M_r \rightarrow \underline{\mathbf{M}}$  be the inverse of  $\omega$  regarded as an  $n$ -ary algebraic partial operation on  $\underline{\mathbf{M}}$ . It follows that  $\sigma(\omega_1(a), \dots, \omega_n(a)) = a$ , for all  $a \in M$ . The partial operation  $\sigma$  on  $\underline{\mathbf{M}}$  is known as the *schizophrenic operation* corresponding to  $\omega_1, \dots, \omega_n$  and was introduced in Davey and Haviar [7].

We illustrate these ideas via the example that was analysed in [11]. Let the algebra  $\underline{\mathbf{D}}$  be the two-element bounded distributive lattice  $\underline{\mathbf{2}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$  and let the algebra  $\underline{\mathbf{M}}$  be the three-element bounded distributive lattice  $\underline{\mathbf{3}} = \langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$  with  $0 < a < 1$ . As  $\underline{\mathbf{3}}$  embeds into  $\underline{\mathbf{2}}^2$ , we see that  $n = 2$  and

we have two endomorphisms  $\omega_1$  and  $\omega_2$  of  $\underline{\mathbf{M}}$ , with range  $\{0, 1\}$ , determined by  $\omega_1(a) := 0$  and  $\omega_2(a) := 1$ . (In [11] these endomorphisms were denoted by  $f$  and  $g$  respectively and the binary partial operation  $\sigma$  was denoted by  $m$ .) Note that the domain,  $M_r$ , of the partial operation  $\sigma$  is the order relation  $\leq$  on  $\underline{\mathbf{D}}$ .

Our main tool in this section will be the following theorem which is proved (slightly more generally) in [12]. The finite-level variant of the theorem is obtained by including the phrases in the square brackets.

**2.5. Theorem.** [12, 4.3] *Let  $\underline{\mathbf{D}}$  be a finite algebra, let  $\underline{\mathbf{M}}$  be a finite algebra in  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{D}})$  which has  $\underline{\mathbf{D}}$  as a subalgebra, let  $\mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}}) = \{\omega_1, \dots, \omega_n\}$  and let  $\sigma : M_r \rightarrow M$  be the schizophrenic operation corresponding to  $\omega_1, \dots, \omega_n$ . Assume that  $\underline{\mathbf{D}} = \langle D; G^D, H^D, R^D, \mathcal{T} \rangle$  is an alter ego of  $\underline{\mathbf{D}}$  that yields a strong duality [at the finite level] on  $\mathcal{A}$  based on  $\underline{\mathbf{D}}$  and let  $\underline{\mathbf{M}} := \langle M; G^M, H^M, R^M, \mathcal{T} \rangle$  be an alter ego of  $\underline{\mathbf{M}}$  that yields a full duality [at the finite level] on  $\mathcal{A}$  based on  $\underline{\mathbf{M}}$ . Assume that, for every non-empty [finite] set  $T$  and every closed substructure  $\mathbf{X}$  of  $\underline{\mathbf{M}}^T$ , the set  $X \cap D^T$  forms a substructure of  $\underline{\mathbf{D}}^T$ . Then the following are equivalent:*

- (1)  $\underline{\mathbf{M}}$  yields a strong duality on  $\mathcal{A}$  [at the finite level];
- (2)  $\underline{\mathbf{M}}$  structurally entails  $M_r$ ;
- (3)  $\sigma$  has an extension in the enriched partial clone of  $\underline{\mathbf{M}}$ .

In order to apply this theorem, we require sufficient conditions on  $\underline{\mathbf{D}}$  and  $\underline{\mathbf{M}}$  for  $\underline{\mathbf{M}}$  to structurally entail the relation  $M_r$ . We achieve this in two stages. Since  $M_r$  is actually an algebraic relation on  $\underline{\mathbf{D}}$ , we first give a simple sufficient condition for  $\underline{\mathbf{D}}$  to structurally entail  $M_r$ . We then show how to lift this from  $\underline{\mathbf{D}}$  up to  $\underline{\mathbf{M}}$ .

**2.6. Lemma.** *Let  $\underline{\mathbf{D}}$  be a finite algebra, let  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{D}})$  and let  $\underline{\mathbf{M}}$  be a finite algebra in  $\mathcal{A}$  that has  $\underline{\mathbf{D}}$  as a subalgebra. Let  $\mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}}) = \{\omega_1, \dots, \omega_n\}$  and let  $M_r$  be the domain of the schizophrenic operation corresponding to  $\omega_1, \dots, \omega_n$ . Assume that  $\underline{\mathbf{D}} = \langle D; G, H, R, \mathcal{T} \rangle$  is an alter ego of  $\underline{\mathbf{D}}$  with respect to which  $e_{\underline{\mathbf{M}}} : \underline{\mathbf{M}} \rightarrow \text{ED}(\underline{\mathbf{M}})$  is an isomorphism. Then  $\underline{\mathbf{D}}$  structurally entails  $M_r$  viewed as a relation on  $D$ .*

*Proof.* By Theorem 2.3, in order to prove that  $\underline{\mathbf{D}}$  structurally entails  $M_r$  it suffices to show that there is a conjunct of atomic formulæ,  $\Phi(v_1, \dots, v_n)$ , in the language of  $\underline{\mathbf{D}}$  such that

$$M_r = \{ (c_1, \dots, c_n) \in D^n \mid \underline{\mathbf{D}} \models \Phi(c_1, \dots, c_n) \}.$$

Since there is only a finite number of maps from  $\mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}})$  to  $D$  that do not preserve  $G \cup H \cup R$ , there is a finite subset  $\Upsilon$  of  $G \cup H \cup R$  such that  $\alpha : \mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}}) \rightarrow D$  preserves  $G \cup H \cup R$  if and only if  $\alpha$  preserves  $\Upsilon$ . For all  $k \in \Upsilon \cup \{=\}$ , let  $\Phi_k(v_1, \dots, v_n)$  be a conjunct of atomic formulæ (involving only the symbol  $k$ ) which describes  $k$  on  $\mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}})$  via the correspondence  $v_i \leftrightarrow \omega_i$ , for all

$i \in \{1, \dots, n\}$ . Consider the following conjunct of atomic formulæ:

$$\Phi(v_1, \dots, v_n) := \bigwedge \{ \Phi_k(v_1, \dots, v_n) \mid k \in \Upsilon \cup \{=\} \}.$$

Since  $\mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}})$  satisfies  $\Phi(\omega_1, \dots, \omega_n)$ , and  $\Phi(v_1, \dots, v_n)$  is a conjunct of atomic formulæ, it follows that  $\Phi(\omega_1(a), \dots, \omega_n(a))$  holds in  $\underline{\mathbf{D}}$  for all  $a \in M$ . Thus,

$$M_r \subseteq \{ (a_1, \dots, a_n) \in D^n \mid \underline{\mathbf{D}} \models \Phi(a_1, \dots, a_n) \}.$$

For the reverse inclusion, let  $(a_1, \dots, a_n) \in D^n$  and assume that  $\underline{\mathbf{D}}$  satisfies  $\Phi(a_1, \dots, a_n)$ . Then the map  $\beta : \mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}}) \rightarrow D$ , given by  $\beta(\omega_i) := a_i$ , for all  $i \in \{1, \dots, n\}$ , is a well defined  $\underline{\mathbf{D}}$ -morphism. Since  $e_{\underline{\mathbf{M}}} : \underline{\mathbf{M}} \rightarrow \text{ED}(\underline{\mathbf{M}})$  is an isomorphism, there exists  $a \in M$  such that  $\beta = e_{\underline{\mathbf{M}}}(a)$ . It follows that  $a_i = \beta(\omega_i) = e_{\underline{\mathbf{M}}}(a)(\omega_i) = \omega_i(a)$ , for all  $i \in \{1, \dots, n\}$ , and hence

$$(a_1, \dots, a_n) = (\omega_1(a), \dots, \omega_n(a)) \in M_r,$$

as required. Since  $M_r$  is determined by a conjunct of atomic formulæ in the language of  $\underline{\mathbf{D}}$ , it follows that  $M_r$  is structurally entailed by  $\underline{\mathbf{D}}$ .  $\square$

In fact, this lemma can be obtained from known results. (An easy calculation shows that, since  $\omega_1, \dots, \omega_n$  is a list of *all* homomorphisms from  $\underline{\mathbf{M}}$  into  $\underline{\mathbf{D}}$ , the relation  $M_r$  is hom-minimal when viewed as a relation on  $D$ . Since, by assumption,  $\underline{\mathbf{D}}$  yields a duality on  $\underline{\mathbf{M}}$  it follows from Corollary 3.4 of [12] that  $\underline{\mathbf{D}}$  structurally entails  $M_r$ .) We have included a direct proof as it is not long and shows explicitly how the required conjunct of atomic formulæ is obtained.

The next result is our principal lemma. The assumptions may look a little technical, but they are just what is required to convert a formula in the language of  $\underline{\mathbf{D}}$  that describes  $M_r$ , viewed as an algebraic relation on  $\underline{\mathbf{D}}$ , into a formula in the language of  $\underline{\mathbf{M}}$  that describes  $M_r$  viewed as an algebraic relation on  $\underline{\mathbf{M}}$ . We shall see that, when combined with Theorem 2.5, this lemma leads immediately to positive answers to Question 2 in the varieties of abelian groups and semilattices.

**2.7. Lemma.** *Let  $\underline{\mathbf{D}}$  be a finite algebra, let  $\mathcal{A} := \mathbb{I}\text{SP}(\underline{\mathbf{D}})$  and let  $\underline{\mathbf{M}}$  be a finite algebra in  $\mathcal{A}$  that has  $\underline{\mathbf{D}}$  as a subalgebra. Let  $\mathcal{A}(\underline{\mathbf{M}}, \underline{\mathbf{D}}) = \{\omega_1, \dots, \omega_n\}$  and let  $M_r$  be the domain of the schizophrenic operation corresponding to  $\omega_1, \dots, \omega_n$ . Assume that  $\underline{\mathbf{D}} = \langle D; G^D, H^D, R^D, \mathcal{T} \rangle$  is an alter ego of  $\underline{\mathbf{D}}$  with respect to which  $e_{\underline{\mathbf{M}}} : \underline{\mathbf{M}} \rightarrow \text{ED}(\underline{\mathbf{M}})$  is an isomorphism, and that  $\underline{\mathbf{M}} = \langle M; G^M, H^M, R^M, \mathcal{T} \rangle$  is an alter ego of  $\underline{\mathbf{M}}$  such that*

- (D) *the unary relation  $D$  on  $M$  is structurally entailed by  $\underline{\mathbf{M}}$ ,*
- (R) *each  $r \in R^D$ , viewed as a relation on  $M$ , is structurally entailed by  $\underline{\mathbf{M}}$ ,*
- (H) *for each  $h \in H^D$ , the set  $\text{dom}(h)$ , regarded as a relation on  $M$ , is structurally entailed by  $\underline{\mathbf{M}}$ , and  $h$  has an extension in  $[G^M \cup H^M]$ ,*
- (G) *each  $g \in G^D$ , viewed as a partial map on  $M$ , has an extension in  $[G^M \cup H^M]$ .*

Then  $\underline{\mathbf{M}}$  structurally entails  $M_r$  viewed as a relation on  $M$ .

*Proof.* By the previous lemma,  $\underline{\mathbf{D}}$  structurally entails  $M_r$  viewed as a relation on  $D$ . Thus, by Theorem 2.3, there is a conjunct of atomic formulæ,  $\Phi(v_1, \dots, v_n)$ , in the language of  $\underline{\mathbf{D}}$  such that

$$M_r = \{ (c_1, \dots, c_n) \in D^n \mid \underline{\mathbf{D}} \models \Phi(c_1, \dots, c_n) \}.$$

We shall show how to use assumptions (D)–(G) in tandem with Theorem 2.3 to convert  $\Phi(v_1, \dots, v_n)$  into a formula  $\Psi(v_1, \dots, v_n)$  in the language of  $\underline{\mathbf{M}}$  such that

$$M_r = \{ (c_1, \dots, c_n) \in M^n \mid \underline{\mathbf{M}} \models \Psi(c_1, \dots, c_n) \}.$$

It then follows from Theorem 2.3, once again, that  $\underline{\mathbf{M}}$  structurally entails  $M_r$  viewed as a relation on  $M$ .

Let  $\Psi_D(v)$  be a conjunct of atomic formulæ in the language of  $\underline{\mathbf{M}}$  such that

$$D = \{ c \in M \mid \underline{\mathbf{M}} \models \Psi_D(c) \}.$$

Define

$$\Psi(v_1, \dots, v_n) := \bigwedge \{ \Psi_D(v_i) \mid i \in \{1, \dots, n\} \} \wedge \Phi'(v_1, \dots, v_n),$$

where  $\Phi'(v_1, \dots, v_n)$  is obtained from  $\Phi(v_1, \dots, v_n)$  via the following replacement rules applied for each  $\ell \geq 0$ .

- Let  $r$  be an  $\ell$ -ary  $R^D$ -relation symbol occurring in the formula  $\Phi(v_1, \dots, v_n)$ . Let  $\Psi_r(v_1, \dots, v_\ell)$  be a formula in the language of  $\underline{\mathbf{M}}$  such that

$$r = \{ (c_1, \dots, c_\ell) \in M^\ell \mid \underline{\mathbf{M}} \models \Psi_r(c_1, \dots, c_\ell) \}.$$

Replace every occurrence of  $r(v_{i_1}, \dots, v_{i_\ell})$  by  $\Psi_r(v_{i_1}, \dots, v_{i_\ell})$ .

- Let  $h$  be an  $\ell$ -ary  $H^D$ -partial-operation symbol occurring in the formula  $\Phi(v_1, \dots, v_n)$ . Let  $\Psi_h(v_1, \dots, v_\ell)$  be a formula in the language of  $\underline{\mathbf{M}}$  with

$$\text{dom}(h) = \{ (c_1, \dots, c_\ell) \in M^\ell \mid \underline{\mathbf{M}} \models \Psi_h(c_1, \dots, c_\ell) \}.$$

Let  $h'$  be an  $\underline{\mathbf{M}}$ -term corresponding to an extension in  $[G^M \cup H^M]$  of the operation  $h \in H^D$ . Replace every occurrence of  $h(v_{i_1}, \dots, v_{i_\ell}) = v_j$  by  $\Psi_h(v_{i_1}, \dots, v_{i_\ell}) \ \& \ h'(v_{i_1}, \dots, v_{i_\ell}) = v_j$ .

- Let  $g$  be an  $\ell$ -ary  $G^D$ -operation symbol occurring in the formula  $\Phi(v_1, \dots, v_n)$ . Let  $g'$  be an  $\underline{\mathbf{M}}$ -term corresponding to an extension in  $[G^M \cup H^M]$  of the operation  $g \in G^D$ . Replace every occurrence of  $g(v_{i_1}, \dots, v_{i_\ell}) = v_j$  by  $g'(v_{i_1}, \dots, v_{i_\ell}) = v_j$ .

As  $\Phi(v_1, \dots, v_n)$  determines  $M_r$  as a relation on  $D$ , it is straightforward to verify that  $\Psi(v_1, \dots, v_n)$  determines  $M_r$  as a relation on  $M$ .  $\square$

We now give several useful theorems that follow from this lemma. Note that, in each case, the assumptions on the algebras  $\underline{\mathbf{D}}$  and  $\underline{\mathbf{M}}$  give  $\text{ISP}(\underline{\mathbf{D}}) = \text{ISP}(\underline{\mathbf{M}})$ , so if  $\underline{\mathbf{D}}$  is strongly dualisable, so is  $\underline{\mathbf{M}}$  (see Hyndman [16]).

**2.8. Theorem.** *Let  $\underline{\mathbf{D}}$  be a finite algebra, let  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{D}})$  and let  $\underline{\mathbf{M}}$  be a finite algebra in  $\mathcal{A}$  that has  $\underline{\mathbf{D}}$  as a subalgebra. Assume that  $\underline{\mathbf{D}} = \langle D; G^D, H^D, R^D, \mathcal{T} \rangle$  strongly dualises  $\underline{\mathbf{D}}$  [at the finite level] and that  $\underline{\mathbf{M}} = \langle M; G^M, H^M, R^M, \mathcal{T} \rangle$  is an alter ego of  $\underline{\mathbf{M}}$  such that*

- (D) *the unary relation  $D$  on  $M$  is structurally entailed by  $\underline{\mathbf{M}}$ ,*
- (R) *each  $r \in R^D$ , viewed as a relation on  $M$ , is structurally entailed by  $\underline{\mathbf{M}}$ ,*
- (H)<sup>b</sup> *for each  $h \in H^D$ , the set  $\text{dom}(h)$ , regarded as a relation on  $M$ , is structurally entailed by  $\underline{\mathbf{M}}$ .*

*If  $\underline{\mathbf{M}}$  fully dualises  $\underline{\mathbf{M}}$  [at the finite level], then  $\underline{\mathbf{M}}$  strongly dualises  $\underline{\mathbf{M}}$  [at the finite level].*

*Proof.* If  $\underline{\mathbf{M}}$  fully dualises  $\underline{\mathbf{M}}$  [at the finite level], then, by Theorem 2.4, every algebraic partial operation on  $\underline{\mathbf{M}}$  whose domain is structurally entailed by  $\underline{\mathbf{M}}$  has an extension in  $[G^M \cup H^M]$ . Hence, assumption (D) guarantees that the operations in  $G^D$ , regarded as partial operations on  $M$ , have an extension in  $[G^M \cup H^M]$  and assumption (H)<sup>b</sup> guarantees that the partial operations in  $H^D$ , regarded as partial operations on  $M$ , also have an extension in  $[G^M \cup H^M]$ . Hence, by the previous lemma,  $\underline{\mathbf{M}}$  structurally entails  $M_r$ . The fact that every map in  $G^D \cup H^D$  has an extension in  $[G^M \cup H^M]$  implies at once that, for every [finite] non-empty set  $T$ , if  $\mathbf{X}$  is a closed substructure of  $\underline{\mathbf{M}}^T$ , then  $X \cap D^T$  forms a closed substructure of  $\underline{\mathbf{D}}^T$ . Hence, by Theorem 2.5, the structure  $\underline{\mathbf{M}}$  strongly dualises  $\underline{\mathbf{M}}$  [at the finite level].  $\square$

Since every algebraic total operation on  $\underline{\mathbf{M}}$  belongs to  $[G^M \cup H^M]$  whenever  $\underline{\mathbf{M}}$  fully dualises  $\underline{\mathbf{M}}$ , the simplest way to guarantee that algebraic relations on  $\underline{\mathbf{D}}$  are structurally entailed by  $\underline{\mathbf{M}}$  is to ask that they are intersections of equalizers of pairs of algebraic total operations on  $\underline{\mathbf{M}}$ . This assumption allows us to remove all mention of the particular nature of  $\underline{\mathbf{M}}$ .

**2.9. Theorem.** *Let  $\underline{\mathbf{D}}$  be a finite algebra, let  $\underline{\mathbf{M}}$  be a finite algebra in  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{D}})$  such that  $\underline{\mathbf{D}}$  is a subalgebra of  $\underline{\mathbf{M}}$ . Assume that  $\underline{\mathbf{D}} = \langle D; G, H, R, \mathcal{T} \rangle$  strongly dualises  $\underline{\mathbf{D}}$  [at the finite level] and that  $D$ , each relation  $r \in R$ , and  $\text{dom}(h)$ , for all  $h \in H$ , is an intersection of equalizers of pairs of algebraic total operations on  $\underline{\mathbf{M}}$ . Then any alter ego  $\underline{\mathbf{M}}$  that fully dualises  $\underline{\mathbf{M}}$  [at the finite level] strongly dualises  $\underline{\mathbf{M}}$  [at the finite level].*

When  $R = \emptyset$  there is a particularly satisfying simplification of this result that involves assumptions on  $\underline{\mathbf{D}}$  only. We say that  $\underline{\mathbf{D}}$  is a *subretract* of  $\underline{\mathbf{M}}$  if  $\underline{\mathbf{D}}$  is a subalgebra of  $\underline{\mathbf{M}}$  and there is a *subretraction* of  $\underline{\mathbf{M}}$  onto  $\underline{\mathbf{D}}$ , that is, a homomorphism  $u : \underline{\mathbf{M}} \rightarrow \underline{\mathbf{D}}$  with  $u|_{\underline{\mathbf{D}}} = \text{id}_{\underline{\mathbf{D}}}$ .

**2.10. Theorem.** *Let  $\underline{\mathbf{D}}$  be a finite algebra and let  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{D}})$ . Assume that  $\underline{\mathbf{D}} = \langle D; G, H, \mathcal{T} \rangle$  strongly dualises  $\underline{\mathbf{D}}$  [at the finite level] and that, for all  $h \in H$ ,*

the set  $\text{dom}(h)$  is an intersection of equalizers of pairs of algebraic total operations on  $\underline{\mathbf{D}}$ . Let  $\underline{\mathbf{M}}$  be a finite algebra in  $\mathcal{A}$  such that  $\underline{\mathbf{D}}$  is a subretract of  $\underline{\mathbf{M}}$ . Then any alter ego  $\underline{\mathbf{M}}$  that fully dualises  $\underline{\mathbf{M}}$  [at the finite level] strongly dualises  $\underline{\mathbf{M}}$  [at the finite level].

*Proof.* Let  $u : \underline{\mathbf{M}} \rightarrow \underline{\mathbf{D}}$  be a homomorphism satisfying  $u \upharpoonright_D = \text{id}_D$ . Then  $D = \text{fix}(u) = \text{eq}^M(u, \text{id}_M)$ . Let  $h \in H$  be  $n$ -ary. By assumption, for some  $m \in \mathbb{N}$ , there are  $n$ -ary algebraic total operations  $g_1, \dots, g_m$  and  $h_1, \dots, h_m$  on  $\underline{\mathbf{D}}$  with

$$\text{dom}(h) = \bigcap_{i=1}^m \text{eq}^D(g_i, h_i).$$

For all  $i \in \{1, \dots, n\}$ , define  $g'_i, h'_i : \underline{\mathbf{M}}^n \rightarrow \underline{\mathbf{M}}$  by  $g'_i := g_i \circ (u \times \dots \times u)$  and  $h'_i := h_i \circ (u \times \dots \times u)$  and let  $\pi_i : \underline{\mathbf{M}}^n \rightarrow \underline{\mathbf{M}}$  be the natural projection. Then,

$$\text{dom}(h) = \bigcap_{i=1}^m \text{eq}^M(g'_i, h'_i) \cap \bigcap_{i=1}^n \text{eq}^M(u \circ \pi_i, \pi_i).$$

Thus the previous theorem applies.  $\square$

Let  $\mathcal{C}$  be a class of algebras and let  $\underline{\mathbf{D}} \in \mathcal{C}$ . Then  $\underline{\mathbf{D}}$  is an *absolute subretract* in  $\mathcal{C}$  if, for all  $\mathbf{A} \in \mathcal{C}$ , whenever  $\underline{\mathbf{D}}$  is a subalgebra of  $\mathbf{A}$  it is actually a subretract of  $\mathbf{A}$ . If  $\underline{\mathbf{D}}$  is an absolute subretract in  $\mathcal{A}_{\text{fin}}$ , then the assumption that  $\underline{\mathbf{D}}$  is a subretract of  $\underline{\mathbf{M}}$  can be replaced by the simpler assumption that  $\underline{\mathbf{D}}$  is a subalgebra of  $\underline{\mathbf{M}}$ . The following simple lemma gives two quite different conditions that guarantee that  $\underline{\mathbf{D}}$  is an absolute subretract in  $\mathcal{A}_{\text{fin}}$ . (In fact, both conditions guarantee that  $\underline{\mathbf{D}}$  is an absolute subretract in  $\mathcal{A}$ , but we won't require this here.)

**2.11. Lemma.** *Let  $\underline{\mathbf{D}}$  be a finite algebra and let  $\mathcal{A} := \mathbb{I}\text{SP}(\underline{\mathbf{D}})$ . If  $\underline{\mathbf{D}}$  is subdirectly irreducible or is injective in  $\mathcal{A}_{\text{fin}}$ , then  $\underline{\mathbf{D}}$  is an absolute subretract in  $\mathcal{A}_{\text{fin}}$ .*

The version of Theorem 2.10 that applies when  $\underline{\mathbf{D}}$  is a total algebra is so striking that it deserves to be stated in its own right.

**2.12. Theorem.** *Let  $\underline{\mathbf{D}}$  be a finite algebra, let  $\mathcal{A} := \mathbb{I}\text{SP}(\underline{\mathbf{D}})$  and let  $\underline{\mathbf{M}}$  be a finite algebra in  $\mathcal{A}$  that has  $\underline{\mathbf{D}}$  as a subalgebra. Assume that  $\underline{\mathbf{D}} = \langle D; G, \mathcal{J} \rangle$  is a total algebra that strongly dualises  $\underline{\mathbf{D}}$  [at the finite level]. Then any alter ego  $\underline{\mathbf{M}}$  of  $\underline{\mathbf{M}}$  that fully dualises  $\underline{\mathbf{M}}$  [at the finite level] strongly dualises  $\underline{\mathbf{M}}$  [at the finite level].*

*Proof.* Since  $\underline{\mathbf{D}}$  is strongly dualised [at the finite level] by a total algebra, it follows that  $\underline{\mathbf{D}}$  is injective in  $\mathcal{A}$  [in  $\mathcal{A}_{\text{fin}}$ ] (see [3, 3.2.10]) and so, by the lemma above,  $\underline{\mathbf{D}}$  is a subretract of  $\underline{\mathbf{M}}$ . Now apply Theorem 2.10.  $\square$

Theorems 2.8–2.12 produce useful results even in the case where  $\underline{\mathbf{M}} = \underline{\mathbf{D}}$ . For example, Theorem 2.9 yields the following special case.

**2.13. Theorem.** *Let  $\underline{\mathbf{D}}$  be a finite algebra. Assume that  $\underline{\mathbf{D}} = \langle D; G, H, R, \mathcal{T} \rangle$  strongly dualises  $\underline{\mathbf{D}}$  [at the finite level] and that each relation  $r \in R$ , and  $\text{dom}(h)$ , for all  $h \in H$ , is an intersection of equalizers of pairs of algebraic total operations on  $\underline{\mathbf{D}}$ . Then any alter ego that fully dualises  $\underline{\mathbf{D}}$  [at the finite level], strongly dualises  $\underline{\mathbf{D}}$  [at the finite level].*

### 3. APPLICATIONS

We now apply Theorem 2.12 to show that Questions 2 and 3 have affirmative answers for the varieties of abelian groups and semilattices.

**Abelian groups.** Let  $\underline{\mathbf{M}} = \langle M; \cdot, ^{-1}, 1 \rangle$  be a finite non-trivial abelian group. Then there is a cyclic subgroup  $\underline{\mathbf{D}}$  of  $\underline{\mathbf{M}}$  such that  $\underline{\mathbf{D}}$  is a direct factor of  $\underline{\mathbf{M}}$  and such that  $\underline{\mathbf{D}}$  and  $\underline{\mathbf{M}}$  generate the same quasi-variety  $\mathcal{A}$ . Since the total algebra  $\underline{\mathbf{D}} = \langle D; \cdot, ^{-1}, 1, \mathcal{T} \rangle$  yields a strong duality on  $\mathcal{A}$  based on  $\underline{\mathbf{D}}$  (see [3, 4.4.2]), we may apply Theorem 2.12 to obtain that every alter ego  $\underline{\mathbf{M}}$  that fully dualises the finite abelian group  $\underline{\mathbf{M}}$  [at the finite level] also strongly dualises  $\underline{\mathbf{M}}$  [at the finite level]. Hence the answers to Questions 2 and 3 for the variety of abelian groups are both ‘yes’.

**Semilattices.** Define  $\underline{\mathbf{S}}_K := \langle \{0, 1\}; \vee, K \rangle$  to be the two-element semilattice with bounds  $K \subseteq \{0, 1\}$ , define  $\mathcal{S}_K := \mathbb{ISP}(\underline{\mathbf{S}}_K)$  and let  $\underline{\mathbf{M}}$  be a finite non-trivial semilattice in  $\mathcal{S}_K$ . We have the following strong dualities on  $\mathcal{S}_K$  based on  $\underline{\mathbf{S}}_K$  given by total algebras.

- (i)  $\underline{\mathbf{S}}_{01} := \langle \{0, 1\}; \vee, 0, 1, \mathcal{T} \rangle$  yields a strong duality on  $\mathcal{S}$  based on the semilattice  $\underline{\mathbf{S}} = \langle \{0, 1\}; \vee \rangle$ .
- (ii)  $\underline{\mathbf{S}}_0 = \langle \{0, 1\}; \vee, 0, \mathcal{T} \rangle$  yields a strong duality on  $\mathcal{S}_0$  based on the semilattice with zero  $\underline{\mathbf{S}}_0 = \langle \{0, 1\}; \vee, 0 \rangle$ .
- (iii)  $\underline{\mathbf{S}}_1 = \langle \{0, 1\}; \vee, 1, \mathcal{T} \rangle$  yields a strong duality on  $\mathcal{S}_1$  based on the semilattice with one  $\underline{\mathbf{S}}_1 = \langle \{0, 1\}; \vee, 1 \rangle$ .
- (iv)  $\underline{\mathbf{S}} = \langle \{0, 1\}; \vee, \mathcal{T} \rangle$  yields a strong duality on  $\mathcal{S}_{01}$  based on the bounded semilattice  $\underline{\mathbf{S}}_{01} = \langle \{0, 1\}; \vee, 0, 1 \rangle$ .

According to Theorem 2.12, if  $\underline{\mathbf{M}}$  is an alter ego of  $\underline{\mathbf{M}}$  that fully dualises the finite semilattice  $\underline{\mathbf{M}}$  [at the finite level], then  $\underline{\mathbf{M}}$  also strongly dualises  $\underline{\mathbf{M}}$  [at the finite level]. So Questions 2 and 3 have affirmative answers for

**Relative-Stone Heyting algebras.** The variety  $\mathcal{L}$  of relative Stone algebras is the subvariety of the variety of Heyting algebras generated by the chains. By an application of Jónsson’s Lemma, the subdirectly irreducible algebras in  $\mathcal{L}$  are precisely the subdirectly irreducible Heyting-algebra chains, that is, the Heyting-algebra chains with a coatom. For all  $n \geq 2$ , let  $\underline{\mathbf{C}}_n := \mathbb{ISP}(\underline{\mathbf{C}}_n)$  be the quasi-variety (in fact, the variety) generated by the  $n$ -element Heyting-algebra chain

$\underline{C}_n = \langle C_n; \wedge, \vee, \rightarrow, 0, 1 \rangle$ , where

$$C_n := \{0, a_1, \dots, a_{n-2}, 1\} \text{ with } 0 = a_0 < a_1 < \dots < a_{n-2} < 1.$$

It follows easily from [3, 7.3.3], that  $\underline{\mathcal{C}}_n := \langle C_n; \text{End}(\underline{C}_n), \text{End}_p(\underline{C}_n), \mathcal{T} \rangle$  yields a strong duality on  $\underline{C}_n$ , where  $\text{End}(\underline{C}_n)$  is the set of all endomorphisms of  $\underline{C}_n$  and  $\text{End}_p(\underline{C}_n)$  is the set of all non-extendible partial endomorphisms of  $\underline{C}_n$ . Since, for  $n \geq 4$ , the type includes partial operations, Theorem 2.12 does not apply in this case. Nevertheless, Theorem 2.10 does. In order to apply Theorem 2.10, we must first prove that the domain of every non-extendible partial endomorphism of  $\underline{C}_n$  is an intersection of equalizers of pairs of endomorphisms of  $\underline{C}_n$ .

Let  $A \subseteq C_n$  be the domain of a non-extendible partial endomorphism  $h$  of  $\underline{C}_n$ . If  $a_{n-2} \notin A$ , we could extend  $h$  by defining  $h(a_{n-2}) = 1$ . Thus,  $a_{n-2} \in A$ . So in order to prove that  $A$  is an intersection of equalizers of pairs of endomorphisms, it suffices to show that, for all  $k \in \{1, \dots, n-3\}$ , the set  $C_n \setminus \{a_k\}$  is such an intersection of equalizers. A map  $g : C_n \rightarrow C_n$  is a non-identity endomorphism of  $\underline{C}_n$  if and only if  $g(0) = 0$ ,  $g^{-1}(1) = \uparrow a_{k+1}$ , for some  $k \in \{0, \dots, n-3\}$ , and  $g \upharpoonright_{\downarrow a_k}$  is an order-embedding of  $\downarrow a_k$  into  $\downarrow a_{n-2}$ . Thus we may define  $g \in \text{End}(\underline{C}_n)$  by

$$\begin{aligned} g(0) &= 0, \quad g(a_1) = a_1, \quad \dots, \quad g(a_{k-1}) = a_{k-1}, \\ g(a_k) &= a_{k+1}, \quad \dots, \quad g(a_{k-3}) = a_{k-2}, \quad g(a_{k-2}) = g(1) = 1. \end{aligned}$$

Then,  $C_n \setminus \{a_k\} = \text{eq}(g^{n-k-1}, g^{n-k-2})$ , as required.

Let  $\underline{\mathbf{M}}$  be a finite non-trivial algebra in  $\mathcal{L}$ . Since  $\underline{\mathbf{M}}$  is a subdirect product of finite chains, it follows that there exists  $n \geq 2$  such that  $\underline{\mathbf{M}} \in \mathcal{C}_n \setminus \mathcal{C}_{n-1}$ . Thus at least one of the chains in the representation of  $\underline{\mathbf{M}}$  as a subdirect product of finite chains is  $\underline{C}_n$ . Hence there is a homomorphism  $u$  of  $\underline{\mathbf{M}}$  onto  $\underline{C}_n$ . Since  $\underline{C}_n$  is projective in  $\mathcal{C}_n$  (see Balbes and Horn [1]), there is an embedding  $v$  of  $\underline{C}_n$  into  $\underline{\mathbf{M}}$  such that  $u \circ v = \text{id}_{C_n}$ . So, without loss of generality, we may assume that  $\underline{C}_n$  is a subretract of  $\underline{\mathbf{M}}$ . It follows from Theorem 2.10 that any alter ego  $\underline{\mathbf{M}}$  that fully dualises  $\underline{\mathbf{M}}$  [at the finite level], strongly dualises  $\underline{\mathbf{M}}$  [at the finite level]. Consequently, Questions 2 and 3 are answered in the affirmative for the variety of relative Stone algebras.

**Bounded distributive lattices.** Let  $\mathcal{D}$  be the variety of bounded distributive lattices. We shall prove the following theorem, thereby showing that Question 2 has an affirmative answer for the variety of bounded distributive lattices.

**3.1. Theorem.** *Let  $\underline{\mathbf{M}}$  be a finite non-trivial bounded distributive lattice. If  $\underline{\mathbf{M}}$  is an alter ego of  $\underline{\mathbf{M}}$  that yields a full duality on  $\mathcal{D}$  (based on  $\underline{\mathbf{M}}$ ), then  $\underline{\mathbf{M}}$  yields a strong duality on  $\mathcal{D}$ .*

Let  $\underline{\mathbf{M}}$  be a finite bounded distributive lattice. The proof breaks up into a very simple argument when  $\underline{\mathbf{M}}$  is a Boolean lattice and a highly non-trivial argument

when  $\underline{\mathbf{M}}$  is non-Boolean. If  $\underline{\mathbf{M}}$  is a Boolean lattice, then it is injective in  $\mathcal{D}$  and consequently every full duality for  $\mathcal{D}$  based on  $\underline{\mathbf{M}}$  is strong by Lemma 2.1.

Now let  $\underline{\mathbf{M}}$  be a finite non-Boolean bounded distributive lattice. Then the two-element bounded distributive lattice  $\underline{\mathbf{2}} = \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$  is a subalgebra of  $\underline{\mathbf{M}}$ . Priestley duality tells us that the alter ego  $\underline{\mathcal{Z}} := \langle \{0, 1\}; \leq, \mathcal{T} \rangle$  yields a strong duality on  $\mathcal{D} = \mathbb{ISP}(\underline{\mathbf{2}})$  (see Priestley [21] or Clark and Davey [3]). Assume that the alter ego  $\underline{\mathcal{M}} := \langle M; G, H, R, \mathcal{T} \rangle$  fully dualises  $\underline{\mathbf{M}}$ . We shall apply Theorem 2.8 to prove that  $\underline{\mathcal{M}}$  strongly dualises  $\underline{\mathbf{M}}$ . As  $\underline{\mathbf{2}}$  is a subretract of  $\underline{\mathbf{M}}$  (via Lemma 2.11, for example), the set  $2 = \{0, 1\}$  is the fixpoint set of an endomorphism of  $\underline{\mathbf{M}}$ . By Theorem 2.4 we know that every endomorphism of  $\underline{\mathbf{M}}$  is in  $[G^M \cup H^M]$ . Thus condition (D) of Theorem 2.8 is satisfied. It remains to establish condition (R) of Theorem 2.8, that is, to prove that the relation

$$\leq_2 = \{(0, 0), (0, 1), (1, 1)\}$$

on  $M$  is structurally entailed by  $\underline{\mathcal{M}}$ . Our proof is similar in structure though not in detail to the corresponding proof in Davey, Haviar and Willard [11].

As Davey, Haviar, Niven and Perkal [8] have given an example of a duality on  $\underline{\mathbf{M}}$  that is full but not strong at the finite level, we are forced to work at the infinite level. To highlight exactly where the infinite level comes in, we shall show that if  $\underline{\mathcal{M}}$  is an alter ego that fully dualises  $\underline{\mathbf{M}}$  at the finite level and is such that  $\varepsilon_{\mathbf{X}} : \mathbf{X} \rightarrow \text{DE}(\mathbf{X})$  is an isomorphism for one specific infinite structure  $\mathbf{X}$  in  $\mathbb{IS}_c\mathbb{P}^+(\underline{\mathcal{M}})$ , then  $\underline{\mathcal{M}}$  structurally entails the relation  $\leq_2$ .

Let  $I := [0, 1] \cap \mathbb{Q}$  and let  $\mathbf{I}$  be the bounded distributive lattice with underlying set  $I$  and with its order inherited from  $\mathbb{R}$ . Let  $a \in M$ . For each real number  $r \in (0, 1)$ , define a homomorphism  $x_r^a \in \mathcal{D}(\mathbf{I}, \underline{\mathbf{M}})$  by

$$x_r^a(q) := \begin{cases} 0, & \text{if } q < r, \\ a, & \text{if } q = r, \\ 1, & \text{if } q > r. \end{cases}$$

Note that if  $r \notin \mathbb{Q}$ , then  $x_r^a \in \mathcal{D}(\mathbf{I}, \underline{\mathbf{2}})$ . The family of all sets of the form

$$U_{i,m} := \{z \in \mathcal{D}(\mathbf{I}, \underline{\mathbf{M}}) \mid z(i) = m\},$$

for  $i \in I$  and  $m \in M$ , is a subbasis for the topology on  $\mathcal{D}(\mathbf{I}, \underline{\mathbf{M}})$  regarded as a subspace of the product space  $M^I$  (with  $M$  endowed with the discrete topology).

**3.2. Lemma.** *Let  $a \in M$ . The set*

$$Y^a := \{x_r^a \mid r \in (0, 1) \cap \mathbb{Q}\} \cup \mathcal{D}(\mathbf{I}, \underline{\mathbf{2}})$$

*is a topologically closed subset of  $\mathcal{D}(\mathbf{I}, \underline{\mathbf{M}})$ .*

*Proof.* Let  $z \in \mathcal{D}(\mathbf{I}, \underline{\mathbf{M}}) \setminus Y^a$ . Then either the image of the map  $z$  is not a subset of  $\{0, a, 1\}$  or there exist  $r, s \in I$  with  $r \neq s$  such that  $z$  maps the closed subinterval

$[r, s] \cap \mathbb{Q}$  of  $I$  onto  $a$ . In the first case,  $z(i) = m$  for some  $i \in I$  and  $m \in M \setminus \{0, a, 1\}$ , which yields  $z \in U_{i,m}$  with  $U_{i,m} \cap Y^a = \emptyset$ . In the second case,  $z \in U_{r,a} \cap U_{s,a}$  with  $(U_{r,a} \cap U_{s,a}) \cap Y^a = \emptyset$  because  $|x^{-1}(a)| \leq 1$  for all  $x \in Y^a$ . Thus  $Y^a$  is a topologically closed subset of  $\mathcal{D}(\mathbf{I}, \mathbf{M})$ .  $\square$

The set  $Y^a$ , with  $a$  chosen outside the centre of  $\mathbf{M}$ , will generate the special structure  $\mathbf{X}$  that we require. The *centre* of  $\mathbf{M}$  is the subset  $C(\mathbf{M})$  consisting of all elements that have a complement in  $\mathbf{M}$ . Since  $\mathbf{M}$  is non-Boolean,  $C(\mathbf{M})$  is a proper subset of  $\mathbf{M}$ . It is well known (see, for example, [9] or [13]) that the following conditions are equivalent:

- (i)  $a \in M \setminus C(\mathbf{M})$ ;
- (ii) there exist prime filters  $F$  and  $G$  of  $\mathbf{M}$  such that  $F \subseteq G$  and  $a \in G \setminus F$ ;
- (iii)  $a \notin \{0, 1\}$  and there exists a subretraction of  $\mathbf{M}$  onto  $\mathbf{3}$ , where  $\mathbf{3}$  is the sublattice of  $\mathbf{M}$  with underlying set  $\mathbf{3} := \{0, a, 1\}$ .

Now fix  $a \in M \setminus C(\mathbf{M})$ . Let  $\varphi_a$  be a retraction of  $\mathbf{M}$  onto  $\mathbf{3}$ . Consider the endomorphisms  $\omega_1$  and  $\omega_2$  of  $\mathbf{3}$  determined by  $\omega_1(a) = 0$  and  $\omega_2(a) = 1$ . Note that  $\omega_1$  and  $\omega_2$  map into  $\{0, 1\}$  and separate the points of  $\mathbf{3}$ . Let  $f$  and  $g$  be the endomorphisms of  $\mathbf{M}$  given by

$$f = \omega_1 \circ \varphi_a \quad \text{and} \quad g = \omega_2 \circ \varphi_a.$$

In the next five lemmas we investigate the special relationship between the set  $Y^a$  and the endomorphisms  $f, g$  and  $\varphi_a$  of  $\mathbf{M}$ .

Note that there are only two elements of  $\mathcal{D}(\mathbf{I}, \mathbf{2})$  that are not in the set  $\{f(x_r^a), g(x_r^a) \mid r \in (0, 1)\}$ , namely  $y^1$ , which maps all but 0 to 1, and  $y^0$ , which maps all but 1 to 0. For notational convenience, we will denote  $y^1$  by  $x_0^a$  and denote  $y^0$  by  $x_1^a$ .

**3.3. Lemma.** *Let  $r \in [0, 1]$  and let  $V$  be an open subset of  $Y^a$ .*

- (a)  $r \in [0, 1] \cap \mathbb{Q}$  &  $f(x_r^a) \in V \implies$   
 $(\exists \delta > 0) (\forall s \in (r, r + \delta)) \ x_s^a, f(x_s^a), g(x_s^a) \in V.$
- (b)  $r \in (0, 1] \cap \mathbb{Q}$  &  $g(x_r^a) \in V \implies$   
 $(\exists \delta > 0) (\forall s \in (r - \delta, r)) \ x_s^a, f(x_s^a), g(x_s^a) \in V.$
- (c)  $r \in [0, 1] \setminus \mathbb{Q}$  &  $x_r^a \in V \implies$   
 $(\exists \delta > 0) (\forall s \in (r - \delta, r + \delta)) \ x_s^a, f(x_s^a), g(x_s^a) \in V.$

*Proof.* (a) Assume  $r \in [0, 1] \cap \mathbb{Q}$  and  $f(x_r^a) \in V$ . Since the topology on  $Y^a$  is the relative topology from  $\mathcal{D}(\mathbf{I}, \mathbf{M})$  and  $V$  is an open subset of  $Y^a$  containing  $f(x_r^a)$ , there exist  $i_1, \dots, i_k, j_1, \dots, j_\ell \in I$  with  $i_1 \leq \dots \leq i_k \leq r < j_1 < \dots < j_\ell$  and

$$f(x_r^a) \in U := (U_{i_1,0} \cap \dots \cap U_{i_k,0} \cap U_{j_1,1} \cap \dots \cap U_{j_\ell,1}) \cap Y^a \subseteq V.$$

Let  $\delta := j_1 - r$ . Then  $s \in (r, r + \delta)$  implies that  $r < s < j_1$ , and therefore  $x_s^a, f(x_s^a), g(x_s^a) \in U \subseteq V$ .

(b) Assume  $r \in (0, 1] \cap \mathbb{Q}$  and  $g(x_r^a) \in V$ . Then there exist  $i_1, \dots, i_k, j_1, \dots, j_\ell \in I$  with  $i_1 < \dots < i_k < r \leq j_1 \leq \dots \leq j_\ell$  and

$$g(x_r^a) \in U := (U_{i_1,0} \cap \dots \cap U_{i_k,0} \cap U_{j_1,1} \cap \dots \cap U_{j_\ell,1}) \cap Y^a \subseteq V,$$

and we choose  $\delta := r - i_k$ .

(c) Assume  $r \in (0, 1) \setminus \mathbb{Q}$  and  $x_r^a \in V$ . Now there exist  $i_1, \dots, i_k, j_1, \dots, j_\ell \in I$  with  $i_1 < \dots < i_k < r < j_1 < \dots < j_\ell$  and

$$x_r^a \in U := (U_{i_1,0} \cap \dots \cap U_{i_k,0} \cap U_{j_1,1} \cap \dots \cap U_{j_\ell,1}) \cap Y^a \subseteq V,$$

and we choose  $\delta := \min(r - i_k, j_1 - r)$ .  $\square$

**3.4. Lemma.** *Let  $r \in (0, 1)$  and assume that  $V$  is an open  $\{f, g\}$ -closed subset of  $Y^a$  with  $x_r^a \in V$ . Then there exists  $\delta > 0$  such that  $x_s^a \in V$  for all  $s \in (r - \delta, r + \delta)$ .*

*Proof.* Since  $V$  is an  $\{f, g\}$ -closed subset of  $Y^a$  we have  $f(x_r^a), g(x_r^a) \in V$ . If  $r \in (0, 1) \cap \mathbb{Q}$ , we choose  $\delta$  to be the minimum of the  $\delta$ 's given by (a) and (b) of Lemma 3.3. If  $r \in (0, 1) \setminus \mathbb{Q}$ , then we use the  $\delta$  given by Lemma 3.3(c).  $\square$

**3.5. Lemma.** *Let  $\alpha : Y^a \rightarrow M$  be a continuous map that preserves  $f$  and  $\varphi_a$ . If  $r \in (0, 1)$  and  $\alpha(x_r^a) \in M \setminus \{0, 1\}$ , then  $\alpha(x_r^a) = a$  and  $r \in (0, 1) \cap \mathbb{Q}$ .*

*Proof.* The unary relations  $2 = \text{fix}(f) = \{0, 1\}$  and  $3 = \text{fix}(\varphi_a) = \{0, a, 1\}$  are structurally entailed by  $f$  and  $\varphi_a$ , respectively. As  $Y^a$  is clearly closed under  $f$  and  $\varphi_a$ , it follows that  $\alpha$  preserves the unary relations 2 and 3. We have  $x_r^a \in 3$  on  $Y^a$ . Thus the assumption that  $\alpha(x_r^a) \in M \setminus \{0, 1\}$  immediately leads to  $\alpha(x_r^a) = a$ . If  $r \in (0, 1) \setminus \mathbb{Q}$ , then  $x_r^a \in 2$  on  $Y^a$ , which yields  $\alpha(x_r^a) \in 2 = \{0, 1\}$ , a contradiction.  $\square$

**3.6. Lemma.** *The set  $Y^a$  is the smallest topologically closed subset of  $M^I$  which is closed under  $f$  and  $g$  and contains the set  $\{x_r^a \mid r \in (0, 1)\}$ .*

*Proof.* Let  $Z$  be the smallest topologically closed and  $\{f, g\}$ -closed subset of  $M^I$  containing the subset  $Y := \{x_r^a \mid r \in (0, 1)\}$  of  $Y^a$ . It is clear that  $Z \subseteq Y^a$  since  $Y^a$  itself is topologically closed and  $\{f, g\}$ -closed. Clearly, the closure of  $Y$  under  $f$  and  $g$  contains every element of  $\mathcal{D}(\mathbf{I}, \mathbf{2})$  except the homomorphisms  $x_1^a$  and  $x_0^a$ . We now prove that  $x_1^a, x_0^a \in \overline{Y}$ .

Let  $x_0^a \in V$ , for some open subset  $V$  of  $Y^a$ . By Lemma 3.3(a), there exists  $\delta > 0$  such that  $x_s^a \in V$ , for all  $s \in (r, r + \delta)$ . Therefore  $V \cap Y \neq \emptyset$ , as required. A similar argument using Lemma 3.3(b) shows that  $x_1^a \in \overline{Y}$ .  $\square$

**3.7. Lemma.** *If  $\alpha : Y^a \rightarrow M$  is continuous and preserves  $f, g$  and  $\varphi_a$ , then there exists  $q \in I$  such that  $\alpha(y) = y(q)$ , for all  $y \in Y^a$ , that is,  $\alpha = \rho_q := \pi_q \upharpoonright_{Y^a}$ .*

*Proof.* It is clear that if  $\alpha = \underline{0}$  then  $\alpha = \rho_0$ , and if  $\alpha = \underline{1}$  then  $\alpha = \rho_1$ . So assume that  $\alpha$  is neither  $\underline{0}$  nor  $\underline{1}$ . Define

$$\mathcal{U}_0 := \{r \in (0, 1) \mid \alpha(x_r^a) = 0\} \quad \text{and} \quad \mathcal{U}_1 := \{r \in (0, 1) \mid \alpha(x_r^a) = 1\}.$$

We will show that  $\mathcal{U}_0$  and  $\mathcal{U}_1$  are open subsets of the interval  $(0, 1)$ .

Firstly, let  $r \in \mathcal{U}_0$ . Then  $\alpha(x_r^a) = 0$  and therefore the set  $V := \alpha^{-1}(\{0\})$  is an  $\{f, g\}$ -closed clopen subset of  $Y^a$  containing  $x_r^a$ . Thus, by Lemma 3.4, there exists  $\delta > 0$  such that  $(r - \delta, r + \delta) \subseteq (0, 1)$  and  $x_s^a \in V$ , for all  $s \in (r - \delta, r + \delta)$ . This yields  $\alpha(x_s^a) = 0$ , for all  $s \in (r - \delta, r + \delta)$ , and therefore  $(r - \delta, r + \delta) \subseteq \mathcal{U}_0$ . Thus  $\mathcal{U}_0$  is an open subset of  $(0, 1)$ , and similarly  $\mathcal{U}_1$  is open by a symmetric argument.

We will now show that, for all  $r \in (0, 1)$ , we have

$$\begin{aligned} \alpha(x_r^a) = a &\implies \\ (\exists \delta > 0) (\forall s \in (r - \delta, r)) (\forall t \in (r, r + \delta)) & [\alpha(x_s^a) = 1 \ \& \ \alpha(x_t^a) = 0]. \quad (*) \end{aligned}$$

Let  $r \in (0, 1)$  and assume that  $\alpha(x_r^a) = a$ . Define the open sets  $V_0 := \alpha^{-1}(\{0\})$  and  $V_1 := \alpha^{-1}(\{1\})$ . Since

$$\alpha(f(x_r^a)) = f(\alpha(x_r^a)) = f(a) = 0,$$

we have  $f(x_r^a) \in V_0$ , and similarly,  $g(x_r^a) \in V_1$ . As  $\alpha(x_r^a) = a$ , Lemma 3.5 implies that  $r \in (0, 1) \cap \mathbb{Q}$ . Hence Lemma 3.3(a) can be applied with  $V = V_0$  and Lemma 3.3(b) can be applied with  $V = V_1$ . Choose  $\delta$  to be the minimum of the  $\delta$ 's given by Lemma 3.3(a) for  $V_0$  and Lemma 3.3(b) for  $V_1$ . Thus, for all  $s \in (r - \delta, r)$ , we have  $x_s^a \in V_1$ , that is,  $\alpha(x_s^a) = 1$ . Similarly, for all  $t \in (r, r + \delta)$ , we obtain  $x_t^a \in V_0$ , that is  $\alpha(x_t^a) = 0$ , as required.

If  $\alpha(x_r^a) \in \{0, 1\}$ , for all  $r \in (0, 1)$ , then  $(0, 1) = \mathcal{U}_0 \cup \mathcal{U}_1$  with  $\mathcal{U}_0 \cap \mathcal{U}_1 = \emptyset$ . But both  $\mathcal{U}_0$  and  $\mathcal{U}_1$  are non-empty (as  $\alpha \notin \{\underline{0}, \underline{1}\}$ ) and open, contradicting the fact that the interval  $(0, 1)$  is connected. Thus there exists some  $q \in (0, 1)$  such that  $\alpha(x_q^a) \notin \{0, 1\}$ . By Lemma 3.5 we conclude that  $\alpha(x_q^a) = a$  and  $q \in (0, 1) \cap \mathbb{Q}$ .

Now define

$$q^\top := \bigvee_{\mathbb{R}} \{r \in (q, 1) \mid \alpha(x_s^a) = 0, \text{ for all } s \in (q, r]\}.$$

From (\*) we know that  $q^\top$  is the join of a non-empty set. Suppose  $q^\top \neq 1$ . If  $\alpha(x_{q^\top}^a) = 1$ , then  $q^\top \in \mathcal{U}_1$ , but then some open neighbourhood of  $q^\top$  belongs to  $\mathcal{U}_1$ . This is a contradiction as  $q^\top$  is the supremum of a subset of  $\mathcal{U}_0$ . Thus,  $\alpha(x_{q^\top}^a) \neq 1$ . It follows from (\*) that we also have  $\alpha(x_{q^\top}^a) \neq a$ . As  $\alpha$  preserves  $\{0, a, 1\} = \text{fix}(\varphi_a)$ , we have  $\alpha(x_{q^\top}^a) \in \{0, a, 1\}$ . Thus  $\alpha(x_{q^\top}^a) = 0$  and so  $q^\top \in \mathcal{U}_0$ . Since  $\mathcal{U}_0$  is open and  $q^\top < 1$ , there exists  $r \in (q^\top, 1)$  such that  $\alpha(x_s^a) = 0$ , for all  $s \in (q^\top, r)$ . Since  $\alpha(x_{q^\top}^a) = 0$ , we conclude that  $\alpha(x_s^a) = 0$ , for all  $s \in (q, r)$ . Choose  $r' \in (q^\top, r)$ . Then  $\alpha(x_s^a) = 0$ , for all  $s \in (q, r']$ . Now the definition of  $q^\top$  yields  $r' \leq q^\top$ , which is a contradiction as  $q^\top < r'$ . Hence  $q^\top = 1$ , and so

$\alpha(x_r^a) = 0$ , for all  $r \in (q, 1)$ . Similarly, it can be shown that  $\alpha(x_r^a) = 1$ , for all  $r \in (0, q)$ .

Thus  $\alpha$  and  $\rho_q$  agree on  $\{x_r^a \mid r \in (0, 1)\}$ . Since both  $\alpha$  and  $\rho_q$  are continuous and preserve  $f$  and  $g$ , it follows that the equalizer  $\text{eq}(\alpha, \rho_q)$  of  $\alpha$  and  $\rho_q$  is a topologically closed and  $\{f, g\}$ -closed subset of  $Y^a$ . Thus,  $\text{eq}(\alpha, \rho_q) = Y^a$ , by Lemma 3.6, whence  $\alpha = \rho_q$ , as claimed.  $\square$

Gathering together the results proved so far, we can now prove our final lemma.

**3.8. Lemma.** *Let  $\underline{\mathbf{M}}$  be a finite non-Boolean bounded distributive lattice. Let  $\mathbf{I}$  be the bounded distributive lattice with underlying set  $[0, 1] \cap \mathbb{Q}$ , let  $a \in M \setminus C(\underline{\mathbf{M}})$  and let  $Y^a$  be the subset of  $\mathcal{D}(\mathbf{I}, \underline{\mathbf{M}})$  defined in Lemma 3.2. Let  $\underline{\mathbf{M}}$  be an alter ego of  $\underline{\mathbf{M}}$  and let  $\mathbf{X}$  be the substructure of  $\mathcal{D}(\mathbf{I})$  generated by  $Y^a$ . Assume that  $\underline{\mathbf{M}}$  yields a full duality on  $\mathcal{D} = \mathbb{ISP}(\underline{\mathbf{M}})$  at the finite level and  $\varepsilon_{\mathbf{X}} : \mathbf{X} \rightarrow \text{DE}(\mathbf{X})$  is an isomorphism. Then  $\underline{\mathbf{M}}$  structurally entails the relation  $\leq_2$ .*

*Proof.* Let  $\underline{\mathbf{M}} := \langle M; G, H, R, \mathcal{T} \rangle$ . Since  $\underline{\mathbf{M}}$  yields a full duality on  $\mathcal{D}$  at the finite level, by Theorem 2.4 we may assume that  $f, g, \varphi_a$  belong to  $G$ .

We first show that the structure  $\mathbf{X}$  is balanced. Let  $\alpha : \mathbf{X} \rightarrow \underline{\mathbf{M}}$  be a morphism. Since  $\alpha|_{Y^a}$  preserves  $f, g, \varphi_a$ , according to Lemma 3.7 there exists  $q \in I$  such that  $\alpha|_{Y^a} = \rho_q^{Y^a}$ , and therefore  $Y^a \subseteq \text{eq}(\alpha, \rho_q^{\mathbf{X}})$ . As  $\alpha$  and  $\rho_q^{\mathbf{X}}$  are morphisms, it follows that  $\text{eq}(\alpha, \rho_q^{\mathbf{X}})$  is a closed substructure of  $\mathbf{X}$ . By definition,  $\mathbf{X}$  is the smallest closed substructure of  $\mathcal{D}(\mathbf{I})$  containing  $Y^a$ , and therefore  $\text{eq}(\alpha, \rho_q^{\mathbf{X}}) = X$ , whence  $\alpha = \rho_q^{\mathbf{X}}$ . If  $p, q \in I$  with  $p \neq q$ , then  $\rho_q^{\mathbf{X}}(x_q^a) = x_q^a(q) = a \neq x_q^a(p) = \rho_p^{\mathbf{X}}(x_q^a)$ . Thus  $\mathbf{X}$  is a balanced substructure of  $\mathcal{D}(\mathbf{I})$ . Since  $\varepsilon_{\mathbf{X}} : \mathbf{X} \rightarrow \text{DE}(\mathbf{X})$  is an isomorphism, by assumption, Lemma 2.2 implies that  $X = \mathcal{D}(\mathbf{I}, \underline{\mathbf{M}})$ .

Define an ordinal sequence  $Y_0^a, Y_1^a, \dots, Y_\sigma^a, \dots$  of subsets of  $\mathcal{D}(\mathbf{I}, \underline{\mathbf{M}})$  by:

$$\begin{aligned} Y_0^a &= Y^a, \\ Y_{\sigma+1}^a &= \overline{[Y_\sigma^a]}, \quad (\text{the topological closure of the } [G \cup H]\text{-closure of } Y_\sigma^a), \\ Y_\lambda^a &= \bigcup_{\sigma < \lambda} Y_\sigma^a, \quad \text{for each limit ordinal } \lambda. \end{aligned}$$

Since  $\mathbf{X}$  is the closed substructure of  $\underline{\mathbf{M}}^I$  generated by  $Y^a$ , it follows that  $X = Y_\kappa^a$ , for some ordinal  $\kappa$  (see [3, Exercise 3.4]). Let  $\kappa$  be the least ordinal such that  $Y_\kappa^a$  contains a homomorphism  $z$  such that  $|z^{-1}(e)| > 1$ , for some  $e \in M \setminus C(\underline{\mathbf{M}})$ . Fix such an  $e$  and denote it by  $e_0$ . As  $X = \mathcal{D}(\mathbf{I}, \underline{\mathbf{M}})$ , such a  $\kappa$  must exist. Since  $|z^{-1}(e_0)| > 1$ , there exist  $r, s \in I$  with  $r < s$  and  $z(r) = z(s) = e_0$ . The minimality of  $\kappa$  and the definition of  $Y_\sigma^a$  guarantee that  $\kappa$  must be a successor ordinal, say  $\kappa = \gamma + 1$ . Thus,  $z \in \overline{[Y_\gamma^a]}$ . If  $z \in \overline{[Y_\gamma^a]} \setminus [Y_\gamma^a]$ , then since  $z \in U_{r, e_0} \cap U_{s, e_0}$  and  $U_{r, e_0} \cap U_{s, e_0}$  is open, there is some homomorphism  $z_1 \in [Y_\gamma^a]$  such that

$z_1 \in U_{r,e_0} \cap U_{s,e_0}$ . Thus,  $|z_1^{-1}(e_0)| > 1$ . Hence, without loss of generality, we may assume that  $z \in [Y_\gamma^a] \setminus Y_\gamma^a$ .

As  $z \in [Y_\gamma^a] \setminus Y_\gamma^a$ , there exists  $n \in \mathbb{N}$ , an  $n$ -ary (partial) operation  $p \in [G \cup H]$  and  $y_1, \dots, y_n \in Y_\gamma^a$  such that  $(y_1, \dots, y_n) \in \text{dom}(p^X)$  and  $p^X(y_1, \dots, y_n) = z \notin Y_\gamma^a$ . By the minimality of  $\gamma$ , we are guaranteed that, for all  $e \in M \setminus C(\underline{\mathbf{M}})$ , each homomorphism  $y_i$  maps at most one element of  $I$  to  $e$ . Hence

$$J := \bigcup \{ y_i^{-1}(e) \mid e \in M \setminus C(\underline{\mathbf{M}}) \ \& \ i \in \{1, \dots, n\} \}$$

is a finite subset of  $I$ . Let  $q \in (I \cap [r, s]) \setminus J$  and define  $b_1 := y_1(q), \dots, b_n := y_n(q)$ . Then  $b_1, \dots, b_n \in C(\underline{\mathbf{M}})$  with  $(b_1, \dots, b_n) \in \text{dom}(p)$  and

$$p(b_1, \dots, b_n) = p(y_1(q), \dots, y_n(q)) = p(y_1, \dots, y_n)(q) = z(q) = e_0.$$

For each  $i \in \{1, \dots, n\}$ , let  $h_i : \underline{\mathbf{M}}^2 \rightarrow \underline{\mathbf{M}}$  be a homomorphism satisfying  $h_i(0, 1) = b_i$ . (Since  $b_i \in C(\underline{\mathbf{M}})$  and therefore has a complement  $b'_i$  in  $\underline{\mathbf{M}}$ , such homomorphisms exist in abundance. For example, first map  $\underline{\mathbf{M}}^2$  onto  $\underline{\mathbf{2}}^2$  so that  $(0, 1)$  in  $\underline{\mathbf{M}}^2$  maps onto  $(0, 1)$  in  $\{0, 1\}^2$ , and then map  $\underline{\mathbf{2}}^2$  isomorphically onto the sublattice of  $\underline{\mathbf{M}}$  with underlying set  $\{0, b_i, b'_i, 1\}$ .) Since  $\underline{\mathbf{M}}$  fully dualises  $\underline{\mathbf{M}}$  at the finite level, we know, by Theorem 2.4 once again, that  $h_i$  is in  $[G \cup H]$ . Now define a (partial) operation  $m \in [G \cup H]$  by

$$m(v_1, v_2) := p(h_1(v_1, v_2), \dots, h_n(v_1, v_2)),$$

where  $\text{dom}(m) := \{ (c_1, c_2) \in M^2 \mid (h_1(c_1, c_2), \dots, h_n(c_1, c_2)) \in \text{dom}(p) \}$ . Then

$$m(0, 0) = p(0, \dots, 0) = 0,$$

$$m(0, 1) = p(h_1(0, 1), \dots, h_n(0, 1)) = p(b_1, \dots, b_n) = e_0,$$

$$m(1, 1) = p(1, \dots, 1) = 1.$$

Note that  $(1, 0) \notin \text{dom}(m)$  as  $m$  is a homomorphism and  $e_0$  has no complement in  $\underline{\mathbf{M}}$ . As the unary relation  $2 = \{0, 1\}$  is structurally entailed by  $\underline{\mathbf{M}}$ , the relation  $\leq_2 = \text{dom}(m) \cap \{0, 1\}^2$  is also structurally entailed by  $\underline{\mathbf{M}}$ .  $\square$

This lemma completes the proof of (the non-Boolean case of) Theorem 3.1.

#### REFERENCES

1. R. Balbes and A. Horn, Injective and projective Heyting algebras, *Trans. Amer. Math. Soc.* **148**, (1970), 549–559.
2. D.M. Clark and B.A. Davey, The quest for strong dualities, *J. Austral. Math. Soc. (Series A)* **58** (1995), 248–280.
3. D.M. Clark and B.A. Davey, *Natural Dualities for the Working Algebraist*, Cambridge University Press, Cambridge, 1998.
4. D.M. Clark and P.H. Krauss, Topological quasivarieties, *Acta Sci. Math. (Szeged)* **47** (1984), 3–39.

5. D.M. Clark, P.M. Idziak, L.R. Sabourin, Cs. Szabó and R. Willard, Natural dualities for quasi-varieties generated by a finite commutative ring, *Algebra Universalis* **46** (2001), 285–320.
6. B.A. Davey, Dualisability in general and endodualisability in particular, in *Logic and Algebra (Pontignano, 1994)*, (A. Ursini and P. Aglianò, eds), Lecture Notes in Pure and Appl. Math. **180**, Dekker, New York, 1996, pp. 437–455.
7. B.A. Davey and M. Haviar, A schizophrenic operation which aids the efficient transfer of strong dualities, *Houston J. Math.* **26** (2000), 215–222.
8. B.A. Davey, M. Haviar, T. Niven and N. Perkal, Full but not strong dualities at the finite level: extending the realm, (submitted).
9. B.A. Davey, M. Haviar and H.A. Priestley, Endoprimal distributive lattices are endodualisable, *Algebra Universalis* **34** (1995), 444–453.
10. B.A. Davey, M. Haviar and H.A. Priestley, The syntax and semantics of entailment in duality theory, *J. Symbolic Logic* **60** (1995), 1087–1114.
11. B.A. Davey, M. Haviar and R. Willard, Full does not imply strong, does it? *Algebra Universalis*, (to appear).
12. B.A. Davey, M. Haviar and R. Willard, Structural entailment, *Algebra Universalis*, (to appear).
13. B.A. Davey and H.A. Priestley, *Introduction to Lattices and Order*, second edition, Cambridge University Press, Cambridge, 2002.
14. B.A. Davey and H. Werner, Dualities and equivalences for varieties of algebras, in *Contributions to Lattice Theory (Szeged, 1980)*, (A.P. Huhn and E.T. Schmidt, eds) Coll. Math. Soc. János Bolyai **33**, North-Holland, Amsterdam, 1983, pp. 101–275.
15. J. Hyndman, Mono-unary algebras are strongly dualizable, *J. Austral. Math. Soc.* **72** (2002), 161–172.
16. J. Hyndman, Strong duality of finite algebras that generate the same quasi-variety, *Algebra Universalis* **51** (2004), 29–34.
17. J. Hyndman and R. Willard, An algebra that is dualizable but not fully dualizable, *J. Pure Appl. Alg.* **151** (2000), 31–42.
18. W.A. Lampe, G.F. McNulty and R. Willard, Full duality among graph algebras and flat graph algebras, *Algebra Universalis* **45** (2001), 311–334.
19. J.G. Pitkethly, Strong and full dualisability: three-element unary algebras, *J. Austral. Math. Soc.* **73** (2002), 187–219.
20. J.G. Pitkethly and B.A. Davey, *Dualisability: unary algebras and beyond*, Springer, 2005.
21. H.A. Priestley, Representation of distributive lattices by means of ordered Stone spaces, *Bull. London Math. Soc.* **2** (1972), 186–190.
22. R. Quackenbush and Cs. Szabó, Strong duality for metacyclic groups, *J. Austral. Math. Soc.* **73** (2002), 377–392.
23. R. Willard, New tools for proving dualizability, in *Dualities, Interpretability and Ordered Structures* (J. Vaz de Carvalho and I. Ferreirim, eds), Centro de Álgebra da Universidade de Lisboa, 1999, pp. 69–74.

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