

## Full but not strong dualities at the finite level: Extending the realm

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ABSTRACT. The realm of natural dualities that are known to be full but not strong at the finite level is a very small one, consisting of a single example. This example, based on the three-element bounded distributive lattice, was presented by Davey, Haviar and Willard [8]. In this paper, we extend this realm to the class of all natural dualities based on a finite non-boolean bounded distributive lattice.

### 1. Introduction

The first two authors together with Ross Willard [8] gave the original and, until now, only example of a finite algebra that has a natural duality which, when restricted to the finite level, is full but not strong. Taking the three-element bounded distributive lattice as the starting algebra, an alter ego was given consisting of the two non-identity endomorphisms and a single binary partial operation. It was proved in [8] that this alter ego yields a full but not strong duality on the class of all *finite* bounded distributive lattices. In stark contrast, it was also proved that any alter ego that yields a full duality on the class of *all* bounded distributive lattices in fact yields a strong duality. The results in [8] raise new questions and open up new research paths within the field of natural dualities. One is immediately led to ask questions such as:

- (a) Could it be that, for a finite algebra that is strongly dualisable, every full duality on the quasi-variety it generates is strong?
- (b) What is it about a finite algebra that allows its full dualities at the finite level to behave so differently from its full dualities at the infinite level?
- (c) Which finite algebras generate a quasi-variety for which every duality that is full [at the finite level] is necessarily strong?
- (d) Which finite algebras have an alter ego that yields a full but not strong duality at the finite level?

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Such musings have already led to some progress. For example, in [6] the first three authors proved that, for each finite abelian group, semilattice and relative-Stone Heyting algebra, every duality that is full [at the finite level] is strong [at the finite level], and, for each finite bounded distributive lattice, every full duality is strong. This provides a partial answer to Question (c) and thereby provides examples with which to study Question (b). While Question (a) might be regarded as wild speculation, it is supported by the limited evidence available to us. In order to make headway on questions such as these, we need a range of examples of finite algebras that possess a full but not strong duality at the finite level. This paper addresses Question (d). We prove the following result:

**Theorem.** *Let  $\underline{\mathbf{M}}$  be a finite non-boolean bounded distributive lattice. Then there is an alter ego  $\underline{\underline{\mathbf{M}}}$  of  $\underline{\mathbf{M}}$  such that*

- (a)  $\underline{\underline{\mathbf{M}}}$  yields a duality that is not full on the class  $\mathcal{D}$  of all bounded distributive lattices, yet
- (b)  $\underline{\underline{\mathbf{M}}}$  yields a duality that is full but not strong on the class of finite bounded distributive lattices.

The authors had hoped to find a conceptual proof of this theorem that would indicate possible generalizations beyond distributive lattices. A natural approach would be to proceed as follows: let  $\underline{\mathbf{M}}$  be a finite non-boolean bounded distributive lattice; then  $\underline{\mathbf{M}}$  has the three-element chain  $\underline{\mathbf{3}}$  as a retract; in [8] an alter ego  $\underline{\underline{\mathbf{3}}}$  for  $\underline{\mathbf{3}}$  was given that yields a full but not strong duality at the finite level; use the retraction from  $\underline{\mathbf{M}}$  onto  $\underline{\mathbf{3}}$  to lift the alter ego  $\underline{\underline{\mathbf{3}}}$  up to an appropriate alter ego  $\underline{\underline{\mathbf{M}}}$  for  $\underline{\mathbf{M}}$ ; *voilà!* Unfortunately, this is too simple minded. We pursued this and many other approaches but to no avail. The hoped-for conceptual proof eluded us and we were left with the direct computational proof presented below. Nevertheless, the result provides us with an infinite number of examples where previously there was only one.

In the following discussion, we attempt to give sufficient detail to provide insight for the novice reader. However, as the alter egos defined here are complex and the proof is rather technical, full understanding requires the reader to have a working knowledge of Priestley duality for finite bounded distributive lattices.

## 2. Some background

A natural duality provides a dual equivalence between a quasi-variety  $\mathcal{A}$  generated by a finite algebra and a class of topological structures  $\mathcal{X}$ . Typical examples covered under this notion are Stone duality [13] for Boolean algebras and Priestley duality [12] for distributive lattices. More precisely, take  $\mathcal{A}$  to be the quasi-variety

generated by some finite algebra  $\underline{\mathbf{M}} := \langle M; F \rangle$ , that is,  $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{M}})$ . In order to set up a natural duality on  $\mathcal{A}$ , we first want to find a topological structure  $\underline{\underline{\mathbf{M}}} = \langle M; G, H, R, \mathcal{T} \rangle$ , called an *alter ego* of  $\underline{\mathbf{M}}$ , that is algebraic over  $\underline{\mathbf{M}}$ . By *algebraic over  $\underline{\mathbf{M}}$*  we mean: each  $n$ -ary element in the set  $G$  (of total operations) is a homomorphism from  $\underline{\mathbf{M}}^n$  to  $\underline{\mathbf{M}}$ , each  $n$ -ary element in the set  $H$  (of partial operations) is a homomorphism from some subalgebra of  $\underline{\mathbf{M}}^n$  to  $\underline{\mathbf{M}}$ , each  $n$ -ary element in the set  $R$  (of relations) is a subalgebra of  $\underline{\mathbf{M}}^n$  and  $\mathcal{T}$  is the discrete topology. Now take  $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+(\underline{\underline{\mathbf{M}}})$  to be the *topological quasi-variety generated by  $\underline{\underline{\mathbf{M}}}$* , that is, all isomorphic copies of closed substructures of non-zero powers of  $\underline{\underline{\mathbf{M}}}$ .

The assumption that the structure in the alter ego is algebraic over  $\underline{\mathbf{M}}$  guarantees that there is a dual adjunction or pre-duality  $\langle D, E, e, \varepsilon \rangle$  between  $\mathcal{A}$  and  $\mathcal{X}$  (see [2, Chapter 1]). The pair of contravariant hom-functors  $D: \mathcal{A} \rightarrow \mathcal{X}$  and  $E: \mathcal{X} \rightarrow \mathcal{A}$  is given on objects by  $D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \underline{\underline{\mathbf{M}}}) \leq \underline{\underline{\mathbf{M}}}^{\mathbf{A}}$ , for all  $\mathbf{A} \in \mathcal{A}$ , and  $E(\mathbf{X}) = \mathcal{X}(\mathbf{X}, \underline{\underline{\mathbf{M}}}) \leq \underline{\underline{\mathbf{M}}}^{\mathbf{X}}$ , for all  $\mathbf{X} \in \mathcal{X}$ . On morphisms,  $D$  and  $E$  are defined by composition: for all  $\mathbf{A}, \mathbf{B} \in \mathcal{A}$  and all homomorphisms  $u: \mathbf{A} \rightarrow \mathbf{B}$ , we have  $D(u): D(\mathbf{B}) \rightarrow D(\mathbf{A})$ , given by  $D(u)(x) = x \circ u$ , and for  $\mathbf{X}, \mathbf{Y} \in \mathcal{X}$  with  $\varphi: \mathbf{X} \rightarrow \mathbf{Y}$ , we have  $E(\varphi): E(\mathbf{Y}) \rightarrow E(\mathbf{X})$ , given by  $E(\varphi)(\alpha) = \alpha \circ \varphi$ . For each  $\mathbf{A} \in \mathcal{A}$ , the evaluation map  $e_{\mathbf{A}}: \mathbf{A} \rightarrow ED(\mathbf{A})$  is defined by  $e_{\mathbf{A}}(a)(x) = x(a)$ , where  $a \in A$  and  $x \in D(\mathbf{A})$ . It follows immediately from the fact that  $\mathcal{A} = \mathbb{ISP}(\underline{\underline{\mathbf{M}}})$  that  $e_{\mathbf{A}}$  is an embedding, for all  $\mathbf{A} \in \mathcal{A}$ . For each  $\mathbf{X} \in \mathcal{X}$ , the embedding  $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow DE(\mathbf{X})$  is defined similarly.

If  $e_{\mathbf{A}}$  is an isomorphism, for each  $\mathbf{A} \in \mathcal{A}$ , then we say that  $\underline{\underline{\mathbf{M}}}$  *yields a duality on  $\mathcal{A}$* , or that  $\underline{\underline{\mathbf{M}}}$  *dualises  $\underline{\mathbf{M}}$* . If  $\underline{\underline{\mathbf{M}}}$  yields a duality on  $\mathcal{A}$  and in addition the morphism  $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow DE(\mathbf{X})$  is an isomorphism for each  $\mathbf{X} \in \mathcal{X}$ , then we say that  $\underline{\underline{\mathbf{M}}}$  *yields a full duality on  $\mathcal{A}$* , or that  $\underline{\underline{\mathbf{M}}}$  *fully dualises  $\underline{\mathbf{M}}$* . Thus, a full duality gives a special kind of dual equivalence between the algebraic quasi-variety  $\mathcal{A}$  and the topological quasi-variety  $\mathcal{X}$ .

In analysing all known full dualities at the time, Davey and Werner [11] noted that they all appeared to satisfy an additional property, namely that  $\underline{\underline{\mathbf{M}}}$  is injective in  $\mathcal{X}$ , and posed the question: “Is this always the case?” Later, Clark and Krauss [3] observed other properties of full dualities. Given a non-empty set  $S$ , they defined a substructure  $\mathbf{X}$  of  $\underline{\underline{\mathbf{M}}}^S$  to be *hom-closed* if, for all sets  $I$  and all subalgebras  $\mathbf{A}$  of  $\underline{\underline{\mathbf{M}}}^I$ , the set  $X$  is closed under every homomorphism from  $\mathbf{A}$  to  $\underline{\underline{\mathbf{M}}}$  interpreted pointwise on  $X$ . Clark and Krauss proved that if  $\underline{\underline{\mathbf{M}}}$  yields a duality on  $\mathcal{A}$ , then  $\underline{\underline{\mathbf{M}}}$  yields a full duality on  $\mathcal{A}$  if and only if every closed substructure  $\mathbf{X}$  of a non-zero power of  $\underline{\underline{\mathbf{M}}}$  is isomorphic to a hom-closed substructure  $\mathbf{Y}$  of a non-zero power of  $\underline{\underline{\mathbf{M}}}$ . (They also introduced the notion of *term-closed subset of  $M^S$*  and proved that it is equivalent to hom-closed, see [2, 3.1.3], but we shall not need this concept here.)

Clark and Krauss observed that every alter ego  $\underline{\mathbf{M}}$  known to yield a full duality had the property that every closed substructure of a non-zero power of  $\underline{\mathbf{M}}$  was actually hom-closed. They posed the question: “Is this always the case?” Later, Clark and Davey [1] showed that this question and the one above from [11] are equivalent. They proved that if  $\underline{\mathbf{M}}$  yields a duality on  $\mathcal{A}$ , then every closed substructure of a non-zero power of  $\underline{\mathbf{M}}$  is hom-closed if and only if  $\underline{\mathbf{M}}$  yields a full duality on  $\mathcal{A}$  and  $\underline{\mathbf{M}}$  is injective in  $\mathcal{X}$ . They coined the phrase *strong duality* to describe full dualities that arise in this way. This paper and [8] give the first examples of alter egos  $\underline{\mathbf{M}}$  that fully dualise  $\underline{\mathbf{M}}$  at the finite level but for which there is a closed substructure  $\mathbf{X}$  of a power of  $\underline{\mathbf{M}}$  and a hom-closed substructure  $\mathbf{Y}$  of a power of  $\underline{\mathbf{M}}$  such that  $\mathbf{X} \cong \mathbf{Y}$ , but  $\mathbf{X} \neq \mathbf{Y}$  (see the paragraph preceding Theorem 8.2).

Now the natural question to ask was: “Are the conditions, full and strong, actually equivalent?” In their text, Clark and Davey present this as the *Full versus Strong Problem* [2, 3.2.7]: *Do there exist a finite algebra  $\underline{\mathbf{M}}$  and a choice of  $\underline{\mathbf{M}}$  such that  $\underline{\mathbf{M}}$  yields a duality on  $\mathcal{A} = \mathbb{ISP}(\underline{\mathbf{M}})$  which is full but not strong, or equivalently, such that  $\underline{\mathbf{M}}$  yields a full duality on  $\mathcal{A}$  with  $\underline{\mathbf{M}}$  non-injective in  $\mathcal{X}$ ?* This remains one of the most tantalising open questions in the foundations of duality theory.

In [8] a partial answer to this question was given by constructing a duality for the class  $\mathcal{D}$  of bounded distributive lattices, based on the three-element chain, that is full but not strong at the finite level. That is, taking  $\underline{\mathbf{M}}$  to be the three-element bounded chain, a non-injective alter ego  $\underline{\mathbf{M}}$  was given which dualises  $\underline{\mathbf{M}}$  and it was proved that  $\varepsilon_{\mathbf{X}}$  is an isomorphism, for every *finite*  $\mathbf{X} \in \mathcal{X}$ . Further it was shown that the duality is not full, and therefore not strong, at the infinite level [8, Theorem 1]. It was also proved that any full duality for the class of bounded distributive lattices, based on the three-element chain, is necessarily strong [8, Theorem 2].

In this paper we show that if  $\underline{\mathbf{M}}$  is a finite non-boolean bounded distributive lattice, then there exists an alter ego  $\underline{\mathbf{M}}$  which yields a full but not strong duality at the finite level on the class  $\mathcal{D}$  of bounded distributive lattices. Our strategy is very close to that used in [8].

While the alter ego  $\underline{\mathbf{M}}$  that we define yields a full duality on the class  $\mathcal{D}_{\text{fin}}$  of finite bounded distributive lattices, it does not yield a full duality on the whole class  $\mathcal{D}$ . Indeed, it is easily seen that the duality given by  $\underline{\mathbf{M}}$  is not strong at the finite level (see Lemma 8.1), but in [6] it is shown that, for each finite non-boolean bounded distributive lattice  $\underline{\mathbf{M}}$ , every full duality for  $\mathcal{D}$  based on  $\underline{\mathbf{M}}$  is necessarily strong. It is impossible to extend our range of examples to a larger class of finite distributive lattices: if  $\underline{\mathbf{M}}$  is a boolean distributive lattice, then it is injective in the class  $\mathcal{D}$  and hence, by the Injectivity Lemma [2, 3.2.10], any alter ego  $\underline{\mathbf{M}}$  of  $\underline{\mathbf{M}}$  that yields a full duality at the finite level, yields a strong duality at the finite level.

### 3. Priestley duality as a tool

Since we are dealing with bounded distributive lattices, it will come as no surprise that we will continue the long tradition of using a natural duality and Priestley duality in tandem; see the discussion in Clark and Davey [2, Chapter 7].

A *Priestley space* is an ordered compact topological space  $\mathbf{Y} = \langle Y; \leq, \mathcal{T} \rangle$  such that, for all  $x, y \in Y$  with  $x \not\leq y$ , there is a clopen decreasing set containing  $y$  but not  $x$ . The class of Priestley spaces equipped with continuous order-preserving maps as morphisms forms a category which is dual to the category of bounded distributive lattices. Priestley's duality [12] is a strong duality between the category  $\mathcal{D} := \mathbb{ISP}(\mathbf{D})$  of bounded distributive lattices and the category  $\mathcal{P} := \mathbb{IS}_c\mathbb{P}^+(\mathbf{D})$  of Priestley spaces, where  $\mathbf{D} := \langle \{0, 1\}; \vee, \wedge, 0, 1 \rangle$  is the two-element bounded lattice and  $\mathbf{D} := \langle \{0, 1\}; \leq, \mathcal{T} \rangle$  is the two-element ordered space with  $0 < 1$  and  $\mathcal{T}$  the discrete topology. We will denote the hom-functors which yield this duality by  $H: \mathcal{D} \rightarrow \mathcal{P}$  and  $K: \mathcal{P} \rightarrow \mathcal{D}$ .

Here and throughout the remainder of the paper, let  $\mathbf{M} := \langle M; \vee, \wedge, 0, 1 \rangle$  be a finite non-boolean bounded distributive lattice. We take the Priestley dual,  $H(\mathbf{M})$ , to be the finite ordered set consisting of  $\mathcal{D}$ -homomorphisms from  $\mathbf{M}$  into  $\mathbf{D}$  ordered pointwise. We will often identify  $\mathbf{M}$  with the lattice of upsets of the ordered set  $H(\mathbf{M})$ . Thus, we shall regularly use the fact that if  $a \in M$  and  $f \in H(\mathbf{M})$ , then  $f(a) = 1 \iff f \in a$  and  $f(a) = 0 \iff f \notin a$ . The elements of  $H(\mathbf{M})$  are called upon to wear many hats. It will be clear from the context whether we are thinking of  $f \in H(\mathbf{M})$  as a homomorphism from  $\mathbf{M}$  to  $\mathbf{D}$ , as an endomorphism of  $\mathbf{M}$ , or as an element of an upset of  $H(\mathbf{M})$ .

In the next section we describe the structure on an alter ego  $\mathbf{M}$  of  $\mathbf{M}$ . The definitions we use will depend upon a fixed enumeration of the maximal chains in  $H(\mathbf{M})$ .

**3.1. Notation.** Let  $C_1, \dots, C_n$  be the pairwise-distinct maximal chains in  $H(\mathbf{M})$ ; note that  $\bigcup_{i=1}^n C_i = H(\mathbf{M})$ . Denote the elements of  $C_i$  by  $f_1^i, \dots, f_{|C_i|}^i$ . If  $|C_i| \geq 2$ , then we assume that  $f_j^i < f_{j+1}^i$ , for all  $1 \leq j \leq |C_i| - 1$ , and if  $|C_i| = 1$ , then we denote the single element of  $C_i$  by either  $f_1^i$  or  $f_{|C_i|}^i$ , as appropriate. For convenience we arrange the chains so that, for  $1 \leq \nu \leq n$ , each of the chains  $C_1, \dots, C_\nu$  contains at least two elements and  $C_{\nu+1}, \dots, C_n$  contain only a single element.

### 4. The alter ego

We consider two sorts of total algebraic operations: the elements of  $H(\mathbf{M})$  regarded as endomorphisms of  $\mathbf{M}$  and certain operations that depend upon the enumeration of the maximal chains in  $H(\mathbf{M})$  given in 3.1 above.

**4.1. The set  $G$  of operations.** For each  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ , define  $\bar{g}_j^i: \mathbf{H}(\underline{\mathbf{M}}) \rightarrow \mathbf{H}(\underline{\mathbf{M}})$  by

$$\bar{g}_j^i(l) := \begin{cases} f_{j+1}^i, & \text{if } l \in \uparrow f_{j+1}^i, \\ f_j^i, & \text{if } l \notin \uparrow f_{j+1}^i, \end{cases} \quad \text{for all } l \in \mathbf{H}(\underline{\mathbf{M}}).$$

Bringing the context back to  $\underline{\mathbf{M}}$ , define  $g_j^i := (e_{\underline{\mathbf{M}}})^{-1} \circ \mathbf{K}(\bar{g}_j^i) \circ e_{\underline{\mathbf{M}}}: \underline{\mathbf{M}} \rightarrow \underline{\mathbf{M}}$ , for all  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ . Thus,  $g_j^i$  is a retraction from  $\underline{\mathbf{M}}$  onto  $\{0, \uparrow f_{j+1}^i, 1\}$  with  $g_j^i(\uparrow f_{j+1}^i) = \uparrow f_{j+1}^i$ . Finally, let

$$G := \{g_j^i \mid 1 \leq i \leq \nu \text{ and } 1 \leq j \leq |C_i| - 1\}.$$

Note that the set  $G$  is determined by the covers in the ordered set  $\mathbf{H}(\underline{\mathbf{M}})$ . Indeed, the statement that  $\underline{\mathbf{M}}$  is non-boolean is equivalent to the assertion that  $G \neq \emptyset$ .

We now define a family  $S$  of partial operations and then define one extra partial operation,  $q$ .

**4.2. The set  $S$  of partial operations.** For  $1 \leq i, i' \leq \nu$ , define  $\text{dom}(s_i^{i'})$  to be the subset of  $M^2$  such that

$$(a, b) \in \text{dom}(s_i^{i'}) \text{ if and only if } f_{|C_i|}^i(a) = f_1^{i'}(b).$$

Clearly, for each pair  $i$  and  $i'$ , the set  $\text{dom}(s_i^{i'})$  is a subalgebra of  $\underline{\mathbf{M}}^2$  as  $f_{|C_i|}^i$  and  $f_1^{i'}$  are homomorphisms. Define  $s_i^{i'}$  on its domain by

$$s_i^{i'}(a, b) := \begin{cases} 1, & \text{if } f_1^{i'} \in a, \\ \text{Max}(\mathbf{H}(\underline{\mathbf{M}})), & \text{if } f_1^{i'} \notin a \text{ and } f_{|C_{i'}|}^{i'} \in b, \\ 0, & \text{if } f_{|C_{i'}|}^{i'} \notin b, \end{cases}$$

where  $\text{Max}(\mathbf{H}(\underline{\mathbf{M}}))$  is the upset of all maximal elements of  $\mathbf{H}(\underline{\mathbf{M}})$ . It is not hard to show that  $s_i^{i'}$  is a homomorphism from  $\text{dom}(s_i^{i'})$  into  $\underline{\mathbf{M}}$ . Now let

$$S := \{s_i^{i'} \mid 1 \leq i, i' \leq \nu\}.$$

(Note that in the definition of  $S$  the case  $i = i'$  is allowed.)

**4.3. The domain of the partial operation  $q$ .** Define  $\text{dom}(q)$  to be the set of  $(|C_1| + \dots + |C_n| - \nu)$ -tuples of elements of  $M$  such that

$$(a_1^1, \dots, a_{|C_1|-1}^1, \dots, a_1^\nu, \dots, a_{|C_\nu|-1}^\nu, a_1^{\nu+1}, \dots, a_1^n) \in \text{dom}(q)$$

if and only if the following hold:

- (A)  $f_{|C_i|}^i(a_j^i) = f_1^i(a_{j+1}^i)$ , for all  $1 \leq i \leq \nu$  with  $|C_i| > 2$  and  $1 \leq j \leq |C_i| - 2$ ;
- (A<sub>2</sub>) if  $|C_i| \leq 2$ , then  $a_1^i$  may be any element of  $M$ ;

(B) if  $f_j^i = f_{j'}^{i'}$ , for some  $1 \leq i, i' \leq \nu$  with  $1 < j < |C_i|$  and  $1 < j' < |C_{i'}|$ , then

$$f_1^i(a_j^i) = f_1^{i'}(a_{j'}^{i'});$$

(C) if  $f_{|C_i|}^i = f_{|C_{i'}|}^{i'}$ , for some  $1 \leq i, i' \leq \nu$ , then  $f_{|C_i|}^i(a_{|C_i|-1}^i) = f_{|C_{i'}|}^{i'}(a_{|C_{i'}|-1}^{i'})$ ;

(D) if  $f_1^i = f_1^{i'}$ , for some  $1 \leq i, i' \leq n$ , then  $f_1^i(a_1^i) = f_1^{i'}(a_1^{i'})$ .

As each  $f_j^i$  is a homomorphism on  $\underline{\mathbf{M}}$ , for all  $1 \leq i \leq n$  and  $1 \leq j \leq |C_i|$ , it is clear that  $\text{dom}(q)$  is a 0, 1-sublattice of  $\underline{\mathbf{M}}^{|C_1|+\dots+|C_n|-\nu}$ . Conditions (A) and (A<sub>2</sub>) are the core components of the definition, with (B), (C) and (D) dealing with any occurrences of overlap between the maximal chains of  $\text{H}(\underline{\mathbf{M}})$ .

We now give an explanation of Condition (A). Assume  $(a_1^1, \dots, a_1^n) \in \text{dom}(q)$ . If  $f_{|C_i|}^i(a_j^i) = f_1^i(a_{j+1}^i)$ , for some  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 2$ , then, viewing  $a_j^i$  and  $a_{j+1}^i$  as upsets of  $\text{H}(\underline{\mathbf{M}})$ , we obtain  $f_{|C_i|}^i \in a_j^i$  if and only if  $f_1^i \in a_{j+1}^i$ . It follows that if  $f_1^i \in a_j^i$  then  $f_1^i \in a_k^i$ , for all  $j \leq k \leq |C_i| - 1$ , and if  $f_1^i \notin a_j^i$  then  $f_1^i \notin a_l^i$ , for all  $1 \leq l \leq j$ . Consider  $a_1^i, \dots, a_{|C_i|-1}^i$  with  $a_j^i \cap C_i \notin \{0, C_i\}$ , for some  $1 \leq j \leq |C_i| - 1$ . Then  $a_j^i$  is the unique element in  $\{a_1^i, \dots, a_{|C_i|-1}^i\}$  such that  $f_{|C_i|}^i \in a_j^i$  and  $f_1^i \notin a_j^i$ . The unique element with this property will be called *proper*. Intuitively, if  $a_j^i$  is proper amongst  $a_1^i, \dots, a_{|C_i|-1}^i$ , for some  $1 \leq i \leq \nu$ , then  $(a_1^i, \dots, a_{|C_i|-1}^i) = (E_1, \dots, E_{j-1}, a_j^i, F_{j+1}, \dots, F_{|C_i|-1})$ , for some upsets  $E_i$  of  $\text{H}(\underline{\mathbf{M}})$  whose intersections with  $C_i$  are empty and some upsets  $F_j$  of  $\text{H}(\underline{\mathbf{M}})$  containing  $C_i$ , where  $a_j^i$  is an upset of  $\text{H}(\underline{\mathbf{M}})$  with  $a_j^i \cap C_i \notin \{0, C_i\}$ .

**4.4. The definition of the partial operation  $q$ .** Given elements  $a_1^i, \dots, a_{|C_i|-1}^i$ , for  $1 \leq i \leq \nu$ , and viewing them as upsets of  $\text{H}(\underline{\mathbf{M}})$ , define

$$\bar{a}_{k_i} := \begin{cases} 0, & \text{if all } a_j^i \text{'s are disjoint from } C_i, \\ \uparrow f_{j+1}^i, & \text{if } a_j^i \text{ is proper for some } 1 \leq j \leq |C_i| - 1, \\ \uparrow f_1^i, & \text{if all } a_j^i \text{'s contain } C_i, \end{cases}$$

where *proper* is defined in the previous paragraph. For  $\nu + 1 \leq i \leq n$ , define  $\bar{a}_{k_i} = C_i$  if  $a_1^i \wedge C_i = C_i$ , otherwise let  $\bar{a}_{k_i} = 0$ . Note that if we are given two proper elements  $a_j^i$  and  $b_{j'}^{i'}$ , from  $a_1^i, \dots, a_{|C_i|-1}^i$  and  $b_1^{i'}, \dots, b_{|C_{i'}|-1}^{i'}$ , respectively, for some  $1 \leq i \leq \nu$ , then we have

$$\bar{a}_{k_i} \vee \bar{b}_{k_{i'}} = \bar{a}_{k_i} \iff j \leq j' \quad \text{and} \quad \bar{a}_{k_i} \wedge \bar{b}_{k_{i'}} = \bar{a}_{k_i} \iff j \geq j'. \quad (*)$$

We now show that if  $f_j^i = f_{j'}^{i'} =: f$ , for  $1 < j < |C_i|$  and  $1 < j' < |C_{i'}|$ , then  $f \in \bar{a}_{k_i}$  if and only if  $f \in \bar{a}_{k_{i'}}$ .

Assume that  $f_j^i = f_{j'}^{i'} =: f \in \bar{a}_{k_i}$ , so that  $\bar{a}_{k_i} \supseteq \uparrow f$  and therefore either  $a_k^i$  is proper, for some  $k \leq j - 1$ , or all  $a_j^i$  contain  $\uparrow f_1^i$ . Also,  $f_{|C_i|}^i(a_{j-1}^i) = f_1^i(a_j^i) = f_1^{i'}(a_{j'}^{i'})$ , by (A) and (B), and therefore  $\uparrow f_1^{i'} \subseteq a_{j'}^{i'}$ . Thus, as a consequence of (A),

the proper element among  $a_1^{i'}, \dots, a_{|C_{i'}|-1}^{i'}$  is  $a_{k'}^{i'}$ , for some  $k' \leq j' - 1$ , so we get  $\bar{a}_{k_{i'}} \supseteq \uparrow f_{j'}^{i'}$  and therefore  $f \in \bar{a}_{k_{i'}}$ . The other direction follows by symmetry.

Conditions (C) and (D) together with (A) and (B) give us the following stronger result.

$$\begin{aligned} \text{If } f_j^i = f_{j'}^{i'} =: f \text{ for some } 1 \leq j \leq |C_i| \text{ and some } 1 \leq j' \leq |C_{i'}|, \text{ then} \\ f \in \bar{a}_{k_i} \iff f \in \bar{a}_{k_{i'}}. \end{aligned} \quad (**)$$

We now define the map  $q$  on its domain by

$$q(a_1^1, \dots, a_{|C_1|-1}^1, \dots, a_1^\nu, \dots, a_{|C_\nu|-1}^\nu, a_1^{\nu+1}, \dots, a_1^n) := \bigvee_{i=1}^n \bar{a}_{k_i}.$$

**4.5. Lemma.** *The map  $q$  is a  $(|C_1| + \dots + |C_n| - \nu)$ -ary algebraic partial operation on  $\underline{\mathbf{M}}$ .*

*Proof.* Since  $\text{dom}(q)$  is a subalgebra of  $\underline{\mathbf{M}}^{|C_1| + \dots + |C_n| - \nu}$ , it remains to prove that  $q$  is a homomorphism. Let

$$\bar{a} := (a_1^1, \dots, a_{|C_1|-1}^1, \dots, a_1^\nu, \dots, a_{|C_\nu|-1}^\nu, a_1^{\nu+1}, \dots, a_1^n) \in \text{dom}(q)$$

and

$$\bar{b} := (b_1^1, \dots, b_{|C_1|-1}^1, \dots, b_1^\nu, \dots, b_{|C_\nu|-1}^\nu, a_1^{\nu+1}, \dots, a_1^n) \in \text{dom}(q).$$

Then

$$q(\bar{a}) \wedge q(\bar{b}) = \left( \bigvee_{i=1}^n \bar{a}_{k_i} \right) \wedge \left( \bigvee_{i=1}^n \bar{b}_{k_i} \right) = \bigvee_{i,i'=1}^n (\bar{a}_{k_i} \wedge \bar{b}_{k_{i'}}).$$

If  $i \neq i'$ , then  $\bar{a}_{k_i} \wedge \bar{b}_{k_{i'}} \in \{0, \uparrow f\}$ , where  $f$  is an element of  $\mathbf{H}(\underline{\mathbf{M}})$  common to both  $C_i$  and  $C_{i'}$  if such an  $f$  exists. If  $\bar{a}_{k_i} \wedge \bar{b}_{k_{i'}} = \uparrow f$ , then  $f \in \bar{a}_{k_i}$  and  $f \in \bar{b}_{k_{i'}}$ . By  $(**)$  we have  $f \in \bar{b}_{k_i}$ , from which it follows that  $\bar{a}_{k_i} \wedge \bar{b}_{k_{i'}} = \bar{a}_{k_i} \wedge \bar{b}_{k_i}$ , and therefore

$$\bigvee_{i,i'=1}^n (\bar{a}_{k_i} \wedge \bar{b}_{k_{i'}}) = \bigvee_{i=1}^n (\bar{a}_{k_i} \wedge \bar{b}_{k_i}).$$

From  $(*)$  it is not hard to see that  $(\bar{a}_{k_i} \wedge \bar{b}_{k_i}) = (\bar{a} \wedge \bar{b})_{k_i}$ , and thus  $q$  preserves  $\wedge$ . Also from  $(*)$  it follows that  $\bar{a}_{k_i} \vee \bar{b}_{k_i} = (\bar{a} \vee \bar{b})_{k_i}$ , and therefore  $q$  also preserves  $\vee$ . It is trivial that  $q$  preserves 0 and 1. Thus  $q$  is a homomorphism.  $\square$

**4.6. Putting the bits together.** Now we can define the alter ego that will be the focus of the remainder of the paper. Let

$$\underline{\mathbf{M}} := \langle M; \mathbf{H}(\underline{\mathbf{M}}) \cup G, S \cup \{q\}, \mathcal{T} \rangle,$$

where  $\mathcal{T}$  is the discrete topology and define  $\mathcal{X} := \mathbb{I}\mathbb{S}_c \mathbb{P}^+(\underline{\mathbf{M}})$ .

Consider the case where  $\underline{\mathbf{M}} = \underline{\mathbf{3}}$ . The set  $\mathbf{H}(\underline{\mathbf{M}})$  contains precisely the elements corresponding to the total operations  $f$  and  $g$  from [8], the set  $G$  contains only the

identity map, the set  $S$  contains just one element which is the partial operation  $h$  defined in [8] and the partial operation  $q$  is the identity map. That is, the alter ego  $\underline{\mathfrak{Z}}$  that we define here, with respect to  $\underline{\mathfrak{Z}}$ , is precisely the alter ego (modulo removing the identity map) defined in [8]. In the case where  $\underline{\mathbf{M}}$  is the four-element chain, the set  $S$  contains exactly one element  $s_1^1$ . In this case, it turns out that  $s_1^1 = g_2^1 \circ q$ . In general, if  $\underline{\mathbf{M}}$  is a finite chain, then  $S = \{s_1^1\}$  and we can define  $s_1^1$  from elements in  $G \cup \{q\}$ , so we do not need the set  $S$  in our alter ego.

As we shall now see, it is an easy consequence of well-known facts about endodualisable bounded distributive lattices that  $\underline{\mathfrak{M}}$  yields a duality.

**4.7. Lemma.** *Let  $\underline{\mathbf{M}}$  be a finite non-boolean bounded distributive lattice and let  $\underline{\mathfrak{M}}$  be the alter ego defined above. Then  $\underline{\mathfrak{M}}$  yields a duality on the class  $\mathfrak{D} = \mathbb{ISP}(\underline{\mathbf{M}})$  of bounded distributive lattices.*

*Proof.* The alter ego  $\underline{\mathfrak{M}}' := \langle M; H(\underline{\mathbf{M}}), \mathcal{J} \rangle$  dualises  $\underline{\mathbf{M}}$ , by Davey, Haviar and Priestley [7]. (This is proved in a more general setting by Davey [4]. See also the discussion of endodualisability in Clark and Davey [2, Chapter 7].) Since enriching the type of a dualising alter ego results in a new dualising alter ego, it follows that  $\underline{\mathfrak{M}}$  also dualises  $\underline{\mathbf{M}}$ . □

### 5. Quasi-equations that hold in $\mathfrak{X}$

We now give a set of quasi-equations that hold in  $\mathfrak{X} := \mathbb{IS}_c\mathbb{P}^+(\underline{\mathfrak{M}})$ . We will use these quasi-equations throughout the remainder of the paper to aid us in the proof of a number of lemmas and the proof of Theorem 8.2. After stating the quasi-equations, we give a brief discussion of their validity. Subsequently, we will simply refer to each quasi-equation by its number in brackets.

Let  $\mathbb{A}$  denote the formal conjunction symbol. The assertion that  $(x_1, \dots, x_n)$  belongs to  $\text{dom}(p)$ , for some partial operation  $p$ , is equivalent to the equation  $p(x_1, \dots, x_n) \approx p(x_1, \dots, x_n)$ . Each of the following is equivalent to finite conjuncts of quasi-equations in the type of  $\underline{\mathfrak{M}}$ .

- (1) For all  $i, i', j, j'$ , with  $0 \leq i, i' \leq n$ ,  $1 \leq j \leq |C_i|$  and  $1 \leq j' \leq |C_{i'}|$ ,

$$f_j^i(f_{j'}^{i'}(x)) \approx f_{j'}^{i'}(x).$$

- (2) For all  $i, i', j, j'$ , with  $0 \leq i, i' \leq n$ ,  $1 \leq j \leq |C_i|$  and  $1 \leq j' \leq |C_{i'}|$ ,

$$f_j^i(x) \approx x \iff f_{j'}^{i'}(x) \approx x.$$

- (3)  $f_1^1(x) \approx f_1^1(y) \mathbb{A} \dots \mathbb{A} f_{|C_n|}^n(x) \approx f_{|C_n|}^n(y) \implies x \approx y$ .

- (4) For all  $i, i', j, j'$ , with  $1 \leq i \leq \nu$ ,  $1 \leq i' \leq n$ ,  $1 \leq j \leq |C_i| - 1$  and  $1 \leq j' \leq |C_{i'}|$ ,

$$f_{j'}^{i'}(g_j^i(x)) \approx \begin{cases} f_{j+1}^i(x) & \text{if } f_{j'}^{i'} \in \uparrow f_{j+1}^i, \\ f_j^i(x) & \text{if } f_{j'}^{i'} \notin \uparrow f_{j+1}^i. \end{cases}$$

- (5) For all  $i, i'$ , with  $1 \leq i, i' \leq n$ ,

$$f_1^i(x) \approx f_{|C_{i'}|}^{i'}(y) \wedge f_{|C_i|}^i(x) \approx f_1^{i'}(y) \implies f_1^i(x) \approx f_{|C_{i'}|}^{i'}(y) \approx f_{|C_i|}^i(x) \approx f_1^{i'}(y).$$

- (6)  $(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n) \in \text{dom}(q) \iff$

$$\begin{aligned} & \bigwedge \{ f_{|C_i|}^i(x_j^i) \approx f_1^i(x_{j+1}^i) \mid 1 \leq i \leq \nu \ \& \ 1 \leq j \leq |C_i| - 2 \} \\ & \wedge \bigwedge \{ f_1^i(x_j^i) \approx f_1^{i'}(x_{j'}^{i'}) \mid 1 \leq i, i' \leq \nu \ \& \ 1 < j < |C_i| \\ & \qquad \qquad \qquad \& \ 1 < j' < |C_{i'}| \ \& \ f_j^i = f_{j'}^{i'} \} \\ & \wedge \bigwedge \{ f_{|C_i|}^i(x_{|C_i|-1}^i) \approx f_{|C_{i'}|}^{i'}(x_{|C_{i'}|-1}^{i'}) \mid 1 \leq i, i' \leq \nu \ \& \ f_{|C_i|}^i = f_{|C_{i'}|}^{i'} \} \\ & \wedge \bigwedge \{ f_1^i(x_1^i) \approx f_1^{i'}(x_1^{i'}) \mid 1 \leq i, i' \leq n \ \& \ f_1^i = f_1^{i'} \}. \end{aligned}$$

- (7) For all  $i, i', j$ , with  $1 \leq i \leq \nu$ ,  $1 \leq j \leq |C_i| - 1$  and  $\nu + 1 \leq i' \leq n$ ,

$$\begin{aligned} & (x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n) \in \text{dom}(q) \implies \\ & \qquad f_j^i(q(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n)) \approx f_1^i(x_j^i) \\ & \qquad \wedge f_{|C_i|}^i(q(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n)) \approx f_{|C_i|}^i(x_{|C_i|-1}^i) \\ & \qquad \wedge f_1^{i'}(q(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n)) \approx f_1^{i'}(x_1^{i'}). \end{aligned}$$

- (8) For all  $i, i'$ , with  $1 \leq i, i' \leq \nu$ ,

$$(x, y) \in \text{dom}(s_i^{i'}) \iff f_{|C_i|}^i(x) \approx f_1^{i'}(y).$$

- (9) For all  $i, i'$ , with  $1 \leq i, i' \leq \nu$ , and all  $f, f' \in \text{H}(\underline{\mathbf{M}}) \setminus \text{Max}(\text{H}(\underline{\mathbf{M}}))$  and  $f'' \in \text{Max}(\text{H}(\underline{\mathbf{M}}))$ ,

$$\begin{aligned} & (x, y) \in \text{dom}(s_i^{i'}) \implies \\ & \qquad f(s_i^{i'}(x, y)) \approx f'(s_i^{i'}(x, y)) \approx f_1^i(x) \wedge f''(s_i^{i'}(x, y)) \approx f_{|C_{i'}|}^{i'}(y). \end{aligned}$$

We now interpret and verify (1)–(9) in  $\underline{\mathbf{M}}$ . Quasi-equation (1) states that  $f'(a)$  is a fixed point of  $f$ , for all  $f, f' \in \text{H}(\underline{\mathbf{M}})$  and all  $a \in M$ . This is clearly true in  $\underline{\mathbf{M}}$  as  $\text{fix}(f) = \{0, 1\}$ , for all  $f \in \text{H}(\underline{\mathbf{M}})$ . It follows that, for all  $f, f' \in \text{H}(\underline{\mathbf{M}})$ , we have  $\text{fix}(f) = \text{fix}(f')$ , which is precisely the interpretation of (2). Quasi-equation (3) states that  $\text{H}(\underline{\mathbf{M}})$  separates points of  $\underline{\mathbf{M}}$ . That is, for any  $a, b \in M$  with  $a \neq b$ , there exists an  $f \in \text{H}(\underline{\mathbf{M}})$  such that  $f(a) \neq f(b)$ . This is clearly true since  $\underline{\mathbf{D}} = \mathbb{I}\text{SP}(\underline{\mathbf{D}})$ .

To see that (4) holds in  $\underline{\mathbf{M}}$ , firstly assume that  $f_{j'}^{i'} \in \uparrow f_{j+1}^i$  and let  $a \in M$ . From the definition of  $g_j^i$  we get  $f_{j'}^{i'}(g_j^i(a)) = (\overline{g}_j^i(f_{j'}^{i'}))(a) = f_{j+1}^i(a)$ . Similarly if  $f_{j'}^{i'} \notin \uparrow f_{j+1}^i$ , then  $f_{j'}^{i'}(g_j^i(a)) = f_j^i(a)$ .

To see that  $\underline{\mathbf{M}}$  satisfies (5), let  $a, b \in M$  and assume that  $f_1^i(a) = f_{|C_{i'}|}^{i'}(b)$  and  $f_{|C_i|}^i(a) = f_1^{i'}(b)$ . We can reinterpret this assumption as  $f_1^i \in a \Leftrightarrow f_{|C_{i'}|}^{i'} \in b$  and  $f_{|C_i|}^i \in a \Leftrightarrow f_1^{i'} \in b$ . If  $f_1^i \in a$ , then  $f_{|C_i|}^i \in a$ . Thus  $f_1^i(a) = f_{|C_i|}^i(a)$  and the conclusion follows. If  $f_1^i \notin a$ , then  $f_{|C_{i'}|}^{i'} \notin b$ . As  $f_{|C_{i'}|}^{i'} \notin b$ , we get  $f_1^{i'} \notin b$ . Thus  $f_{|C_{i'}|}^{i'}(b) = f_1^{i'}(b)$  and the conclusion follows.

Certainly  $\underline{\mathbf{M}}$  satisfies (6) as it is just the definition of the domain of the partial operation  $q$ .

To see that  $\underline{\mathbf{M}}$  satisfies (7), let  $(a_1^1, \dots, a_{|C_\nu|-1}^\nu, \dots, a_1^n) \in \text{dom}(q)$ , let  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ . We first want to show that

$$f_j^i \in \bigvee_{i'=1}^n \overline{a}_{k_{i'}} \iff f_1^i \in a_j^i.$$

If  $f_j^i \in \bigvee_{i'=1}^n \overline{a}_{k_{i'}}$ , then by (\*\*) we know that  $f_j^i \in \overline{a}_{k_i}$ . This can only happen if  $\overline{a}_{k_i} = \uparrow f_1^i$  or  $a_l^i$  is proper for some  $1 \leq l \leq j - 1$ , by the definition of the  $\overline{a}_{k_i}$ 's. In either case,  $f_1^i \in a_j^i$ . If  $f_j^i \notin \bigvee_{i'=1}^n \overline{a}_{k_{i'}}$ , then  $\overline{a}_{k_i} = \uparrow f_{l+1}^i$ , for some  $j \leq l \leq |C_i| - 1$  or  $\overline{a}_{k_i} = 0$ . If  $\overline{a}_{k_i} = 0$ , then  $a_j^i$  is disjoint from  $\uparrow f_1^i$  and therefore  $f_1^i \notin a_j^i$ . If  $\overline{a}_{k_i} = \uparrow f_{l+1}^i$ , for  $j \leq l$ , then  $a_l^i$  is proper and therefore  $f_1^i \notin a_j^i$ . The remaining parts of (7) can be shown in a similar way.

Quasi-equation (8) is just the definition of the domain of  $s_i^{i'}$ . Quasi-equation (9) describes how non-maximal and maximal elements operate on  $s_i^{i'}$ , and is easy to prove.

## 6. The mimicking order

In Section 8 we shall prove that  $\underline{\mathbf{M}}$  fully dualises  $\underline{\mathbf{M}}$  at the finite level. In order to do so, we shall associate an ordered boolean space,  $\mathbf{P}_X$ , to every structure  $\mathbf{X}$  in  $\mathfrak{X}$ . We will then show that  $\varepsilon_X : \mathbf{X} \rightarrow \text{DE}(\mathbf{X})$  is an isomorphism if and only if  $\mathbf{P}_X$  is a Priestley space. Since every finite ordered boolean space is a Priestley space, this will complete the proof.

For each  $\mathbf{X} \in \mathfrak{X}$ , define the subset  $P_X$  of  $X$  to be the fixpoint set of  $X$  under any  $f \in \text{H}(\underline{\mathbf{M}})$ . As  $\text{fix}(f) = \text{fix}(f')$ , for all  $f, f' \in \text{H}(\underline{\mathbf{M}})$ , the choice of  $f$  is irrelevant. Define a binary relation  $\preceq$  on  $P_X$  by  $u \preceq v$  if and only if

$$(\exists x \in X)(\exists i \in \{1, \dots, n\}) f_1^i(x) = u \wedge f_{|C_i|}^i(x) = v.$$

If such an  $x \in X$  and  $i \in \{1, \dots, n\}$  exist for some pair  $u, v \in P_X$ , we will say that  $x$  witnesses  $u \preceq v$  at  $i$ . Define  $\mathbf{P}_X := \langle P_X; \preceq, \mathcal{T} \rangle$ , where  $\mathcal{T}$  is the subspace topology.

It is important to note that, when  $\mathbf{X}$  is a closed substructure of a power of  $\underline{\mathbf{M}}$ , the order  $\preceq$  defined on  $P_X$  need not be the pointwise order.

**6.1. Lemma.** *Let  $\mathbf{X} \in \mathfrak{X}$  and define  $\mathbf{P}_X$  as above.*

- (a) *The topological space  $\langle P_X; \mathcal{T} \rangle$  is boolean.*
- (b) *The relation  $\preceq$  is topologically closed.*
- (c) *The relation  $\preceq$  is an order.*

*Proof.* Since the topology on  $\mathbf{X}$  is Hausdorff and each  $f \in \mathbf{H}(\underline{\mathbf{M}})$  is continuous,  $P_X = \text{fix}(f)$  is a closed subset of  $X$ . Since  $\langle X; \mathcal{T} \rangle$  is compact, the subspace  $\langle P_X; \mathcal{T} \rangle$  is compact and therefore boolean. Note that

$$\preceq = \bigcup_{i=1}^n (f_1^i \sqcap f_{|C_i|}^i)(X) = \{ (f_1^i(x), f_{|C_i|}^i(x)) \mid x \in X \text{ and } 1 \leq i \leq n \},$$

where  $f_1^i \sqcap f_{|C_i|}^i : X \rightarrow X \times X$  is the natural map. As the topology on  $\mathbf{X}$  is compact, and since  $f_1^i \sqcap f_{|C_i|}^i$  is continuous, the set  $\preceq$  is closed in  $X \times X$  and is therefore closed in  $P_X \times P_X$ . This proves (a) and (b).

Let  $u \in P_X$ . By (2),  $f_1^i(u) = u = f_{|C_i|}^i(u)$ , for all  $1 \leq i \leq n$ , whence  $u \preceq u$ . Hence, the relation  $\preceq$  is reflexive. Now let  $u, v \in P_X$  with  $u \preceq v$  and  $v \preceq u$ . There exists a witness  $x$  at  $i$  and a witness  $y$  at  $i'$ , for some  $1 \leq i, i' \leq n$ , such that  $f_1^i(x) = u = f_{|C_{i'}|}^{i'}(y)$  and  $f_1^{i'}(y) = v = f_{|C_i|}^i(x)$ . By (5), we have  $f_1^i(x) = f_1^{i'}(y) = f_{|C_i|}^i(x) = f_{|C_{i'}|}^{i'}(y)$  and thus  $u = v$ . Hence the relation  $\preceq$  is anti-symmetric.

Finally, let  $u, v, w \in P_X$  with  $u \preceq v \preceq w$ . There exist  $x, y \in X$  with  $f_1^i(x) = u$  and  $f_{|C_i|}^i(x) = v = f_1^{i'}(y)$  and  $f_{|C_{i'}|}^{i'}(y) = w$ , for some  $1 \leq i, i' \leq n$  (with the possibility that  $i = i'$ ). If  $i > \nu$  or  $i' > \nu$ , then  $u = v$  or  $v = w$  and we are done. So assume  $i, i' \leq \nu$ . By (8) we get  $(x, y) \in \text{dom}(s_i^{i'})$ . From (9) we get  $f_{|C_{i'}|}^{i'}(s_i^{i'}(x, y)) = f_{|C_{i'}|}^{i'}(y) = w$  and  $f_1^{i'}(s_i^{i'}(x, y)) = f_1^i(x) = u$ , and thus  $s_i^{i'}(x, y)$  witnesses  $u \preceq w$  at  $i'$  by (9). (Note that  $s_i^{i'}(x, y)$  also witnesses  $u \preceq w$  at any  $1 \leq i'' \leq \nu$  by (9).) Thus the relation  $\preceq$  is transitive. We have now proved (c).  $\square$

We shall require the following instance of the order relation.

**6.2. Lemma.** *Let  $\mathbf{X} \in \mathfrak{X}$ . For all  $f, f' \in \mathbf{H}(\underline{\mathbf{M}})$  with  $f < f'$ , and all  $x \in X$ , we have  $f(x) \preceq f'(x)$ .*

*Proof.* By (4) it is clear that  $g_j^i(x)$  witnesses  $f_j^i(x) \preceq f_{j+1}^i(x)$  at  $i$ , and so the result follows by transitivity and the fact that if  $f'$  covers  $f$  in  $\mathbf{H}(\underline{\mathbf{M}})$ , then there is an  $1 \leq i \leq \nu$  such that  $f, f' \in C_i$ .  $\square$

## 7. The schizophrenic partial operation

There is a family of algebraic partial operations on  $\underline{\mathbf{M}}$ , known as schizophrenic partial operations. If any one of these partial operations were in the type  $\underline{\mathbf{M}}$ , it would force our full duality to be strong (see Davey and Haviar [5] and Davey, Haviar and Willard [9]). While we have constructed  $\underline{\mathbf{M}}$  so as to avoid these partial operations, the relationship between a particular schizophrenic partial operation and the partial operations in the type of  $\underline{\mathbf{M}}$  will play a vital role in our proof. For each natural number  $\ell$  and every family of  $\ell$  homomorphisms from  $\underline{\mathbf{M}}$  into  $\underline{\mathbf{D}}$  that separate the points of  $M$ , there is a corresponding  $\ell$ -ary schizophrenic partial operation on  $\underline{\mathbf{M}}$ . We now define the one that is intimately connected to the alter ego  $\underline{\mathbf{M}}$ .

Define a  $(|C_1| + \dots + |C_n|)$ -ary algebraic partial operation  $\sigma$  on  $\underline{\mathbf{M}}$  as follows: the domain of  $\sigma$  is given by

$$\text{dom}(\sigma) := \{ (f_1^1(a), \dots, f_{|C_n|}^n(a)) \mid a \in M \},$$

and on its domain,  $\sigma$  is defined by

$$\sigma(f_1^1(a), \dots, f_{|C_n|}^n(a)) := a.$$

In general,  $\sigma$  will have some repeated co-ordinates in its domain. This will not be the case when  $\underline{\mathbf{M}}$  is a product of chains and therefore  $\text{H}(\underline{\mathbf{M}})$  is a disjoint union of chains. We refer to [5] for a discussion of the role of the schizophrenic partial operation  $\sigma$  in creating strong dualities.

We now show how the partial operation  $\sigma$  relates to the alter ego  $\underline{\mathbf{M}}$ . While these results hold *in*  $M$ , they make no sense in arbitrary structures in  $\mathfrak{X}$  as  $\sigma$  is not in the type of  $\underline{\mathbf{M}}$ .

**7.1. Lemma.** *The following hold in  $\underline{\mathbf{M}}$ .*

- (a) *If  $(a_1^1, \dots, a_{|C_n|}^n) \in \text{dom}(\sigma)$ , then  $f_j^i(\sigma(a_1^1, \dots, a_{|C_n|}^n)) = a_j^i$ , for all  $i, j$  with  $1 \leq i \leq n$  and  $1 \leq j \leq |C_i|$ .*
- (b) *If  $(a_1^1, \dots, a_{|C_1|-1}^1, \dots, a_1^\nu, \dots, a_{|C_\nu|-1}^\nu, a_1^{\nu+1}, \dots, a_1^n) \in \text{dom}(\sigma)$ , then*

$$\begin{aligned} & (f_1^1(a_1^1), \dots, f_1^1(a_{|C_1|-1}^1), f_{|C_1|}^1(a_{|C_1|-1}^1), \dots, \\ & \quad f_1^\nu(a_1^\nu), \dots, f_1^\nu(a_{|C_\nu|-1}^\nu), f_{|C_\nu|}^\nu(a_{|C_\nu|-1}^\nu), \\ & \quad f_1^{\nu+1}(a_1^{\nu+1}), \dots, f_1^n(a_1^n)) \in \text{dom}(\sigma) \end{aligned}$$

and

$$\begin{aligned} & \sigma(f_1^1(a_1^1), \dots, f_1^1(a_{|C_1|-1}^1), f_{|C_1|}^1(a_{|C_1|-1}^1), \dots, \\ & \quad f_1^\nu(a_1^\nu), \dots, f_1^\nu(a_{|C_\nu|-1}^\nu), f_{|C_\nu|}^\nu(a_{|C_\nu|-1}^\nu), \\ & \quad \quad \quad f_1^{\nu+1}(a_1^{\nu+1}), \dots, f_1^n(a_1^n)) \\ & = q(a_1^1, \dots, a_{|C_1|-1}^1, \dots, a_1^\nu, \dots, a_{|C_\nu|-1}^\nu, a_1^{\nu+1}, \dots, a_1^n). \end{aligned}$$

(c) If  $(a_1^1, \dots, a_{|C_n|}^n) \in \text{dom}(\sigma)$ , then, for  $i, j$  with  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ , we have  $g_j^i(\sigma(a_1^1, \dots, a_{|C_n|}^n)) = \sigma(b_1^1, \dots, b_{|C_n|}^n)$ , where

$$b_{j'}^{i'} = \begin{cases} a_{j+1}^i, & \text{if } f_{j'}^{i'} \in \uparrow f_{j+1}^i, \\ a_j^i, & \text{if } f_{j'}^{i'} \notin \uparrow f_{j+1}^i, \end{cases}$$

for all  $i', j'$  with  $1 \leq i' \leq n$  and  $1 \leq j' \leq |C_{i'}|$ .

(d) If  $(a, b) \in \text{dom}(s_i^{i'})$ , for some  $i, i'$  with  $1 \leq i, i' \leq \nu$ , then

$$\begin{aligned} & (\underbrace{f_1^i(a), \dots, f_1^i(a)}_{|C_1|-1}, \underbrace{f_{|C_{i'}|}^{i'}(b), \dots, f_{|C_{i'}|}^{i'}(b)}_{|C_\nu|-1}, \underbrace{f_1^i(a), \dots, f_1^i(a)}_{|C_\nu|-1}, \underbrace{f_{|C_{i'}|}^{i'}(b), \dots, f_{|C_{i'}|}^{i'}(b)}_{n-\nu}) \in \text{dom}(\sigma) \end{aligned}$$

and

$$\begin{aligned} s_i^{i'}(a, b) = \sigma & (\underbrace{f_1^i(a), \dots, f_1^i(a)}_{|C_1|-1}, \underbrace{f_{|C_{i'}|}^{i'}(b), \dots, f_{|C_{i'}|}^{i'}(b)}_{|C_\nu|-1}, \underbrace{f_1^i(a), \dots, f_1^i(a)}_{|C_\nu|-1}, \\ & \quad \quad \quad \underbrace{f_{|C_{i'}|}^{i'}(b), f_{|C_{i'}|}^{i'}(b), \dots, f_{|C_{i'}|}^{i'}(b)}_{n-\nu}). \end{aligned}$$

*Proof.* Claim (a) is immediate using the definition of  $\sigma$ .

(b) Let  $(a_1^1, \dots, a_{|C_1|-1}^1, \dots, a_1^\nu, \dots, a_{|C_\nu|-1}^\nu, a_1^{\nu+1}, \dots, a_1^n) \in \text{dom}(q)$ . Then by (7) we have

$$\begin{aligned} & (f_1^1(a_1^1), \dots, f_1^1(a_{|C_1|-1}^1), f_{|C_1|}^1(a_{|C_1|-1}^1), \dots, \\ & \quad f_1^\nu(a_1^\nu), \dots, f_1^\nu(a_{|C_\nu|-1}^\nu), f_{|C_\nu|}^\nu(a_{|C_\nu|-1}^\nu), f_1^{\nu+1}(a_1^{\nu+1}), \dots, f_1^n(a_1^n)) \\ & = (f_1^1(q(a_1^1, \dots, a_1^n)), \dots, f_{|C_1|-1}^1(q(a_1^1, \dots, a_1^n)), f_{|C_1|}^1(q(a_1^1, \dots, a_1^n)), \dots, \\ & \quad f_1^\nu(q(a_1^1, \dots, a_1^n)), \dots, f_{|C_\nu|-1}^\nu(q(a_1^1, \dots, a_1^n)), f_{|C_\nu|}^\nu(q(a_1^1, \dots, a_1^n)), \\ & \quad \quad \quad f_1^{\nu+1}(q(a_1^1, \dots, a_1^n)), \dots, f_1^n(q(a_1^1, \dots, a_1^n))) \in \text{dom}(\sigma), \end{aligned}$$

and the result follows from the definition of  $\sigma$ .

(c) Let  $(a_1^1, \dots, a_{|C_n|}^n) \in \text{dom}(\sigma)$ . This gives  $(a_1^1, \dots, a_{|C_n|}^n) = (f_1^1(a), \dots, f_{|C_n|}^n(a))$  for some  $a \in M$ , and therefore  $\sigma(a_1^1, \dots, a_{|C_n|}^n) = a$ . Thus, for  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ ,

$$\begin{aligned} g_j^i(\sigma(a_1^1, \dots, a_{|C_n|}^n)) &= g_j^i(a) \\ &= \sigma(f_1^1(g_j^i(a)), \dots, f_{|C_n|}^n(g_j^i(a))) \\ &= \sigma(b_1^1, \dots, b_{|C_n|}^n) \qquad \text{by (4).} \end{aligned}$$

(d) Let  $(a, b) \in \text{dom}(s_i^{i'})$ , for some  $i, i'$  with  $1 \leq i, i' \leq \nu$ . We show that the  $(|C_1| + \dots + |C_n|)$ -tuple in (d) is  $(f_1^1(c), \dots, f_{|C_n|}^n(c))$ , where  $c = s_i^{i'}(a, b)$ . To prove this, as well as the equality in (d), we must show that  $f(s_i^{i'}(a, b)) = f_1^i(a)$ , for all  $f \in \text{H}(\underline{\mathbf{M}}) \setminus \text{Max}(\text{H}(\underline{\mathbf{M}}))$ , and that  $f(s_i^{i'}(a, b)) = f_{|C_{i'}|}^{i'}(b)$ , for all  $f \in \text{Max}(\text{H}(\underline{\mathbf{M}}))$ . This is exactly (9).  $\square$

**8. Full but not strong duality at the finite level**

We are now in a position to prove the major result of this paper, namely that  $\underline{\mathbf{M}}$  yields a full but not strong duality at the finite level. We first prove that  $\underline{\mathbf{M}}$  does not yield a strong duality at the finite level. To do this, it suffices to find a substructure of a finite power of  $\underline{\mathbf{M}}$  that is not closed under some algebraic partial operation and so is not hom-closed (see Clark and Davey [2, Chapter 3]). Denote the centre of  $\underline{\mathbf{M}}$  by  $C(\underline{\mathbf{M}})$ , that is,  $C(\underline{\mathbf{M}})$  is the collection of all elements of  $M$  which have a complement in  $\underline{\mathbf{M}}$ .

**8.1. Lemma.** *The centre  $C(\underline{\mathbf{M}})$  of  $\underline{\mathbf{M}}$  is a substructure of  $\underline{\mathbf{M}}$  that is not closed under the algebraic partial operation  $\sigma$ . Consequently, the alter ego  $\underline{\mathbf{M}}$  does not yield a strong duality at the finite level.*

*Proof.* It is easy to see that the centre of  $\underline{\mathbf{M}}$  is not closed under  $\sigma$ : just take some  $a \in M \setminus C(\underline{\mathbf{M}})$ , which exists as  $\underline{\mathbf{M}}$  is non-boolean. Then  $(f_1^1(a), \dots, f_{|C_n|}^n(a))$  belongs to the domain of  $\sigma$  on  $C(\underline{\mathbf{M}})$ , but  $\sigma(f_1^1(a), \dots, f_{|C_n|}^n(a)) = a \notin C(\underline{\mathbf{M}})$ .

Recall that a 0, 1-lattice homomorphism preserves existing complements. It follows that  $C(\underline{\mathbf{M}})$  is closed under an algebraic partial operation  $k: \mathbf{A} \rightarrow \underline{\mathbf{M}}$ , where  $\mathbf{A} \leq \underline{\mathbf{M}}^n$  for some  $n \in \mathbb{N}$ , provided

$$a_1, \dots, a_n \in C(\underline{\mathbf{M}}) \ \& \ (a_1, \dots, a_n) \in \text{dom}(k) \implies (c(a_1), \dots, c(a_n)) \in \text{dom}(k),$$

where  $c(a)$  denotes the complement of  $a$  in  $C(\underline{\mathbf{M}})$ . It follows immediately that  $C(\underline{\mathbf{M}})$  is closed under the total operations in  $\text{H}(\underline{\mathbf{M}}) \cup G$ . Now take some  $s_i^{i'} \in S$ . Thus  $(a, b) \in \text{dom}(s_i^{i'})$  if and only if  $(f_{|C_i|}^i \in a \Leftrightarrow f_1^{i'} \in b)$ , and this clearly implies  $(f_{|C_i|}^i \in c(a) \Leftrightarrow f_1^{i'} \in c(b))$ , whence  $(c(a), c(b)) \in \text{dom}(s_i^{i'})$ . A similar argument can be given for  $q$ , using (A),(A<sub>2</sub>),(B),(C) and (D) from 4.3. Thus  $C(\underline{\mathbf{M}})$  is a

substructure of  $\underline{\mathbf{M}}$ , but is not closed under  $\sigma$ . Hence  $\underline{\mathbf{M}}$  does not yield a strong duality at the finite level.  $\square$

In Section 2 we promised to give an example of a (closed) substructure  $\mathbf{X}$  of a power of  $\underline{\mathbf{M}}$  that is isomorphic to a hom-closed substructure  $\mathbf{Y}$  of a power of  $\underline{\mathbf{M}}$  with  $\mathbf{X} \neq \mathbf{Y}$ . The previous lemma together with the next theorem show that, for any finite non-boolean distributive lattice  $\underline{\mathbf{M}}$ , the centre  $C(\underline{\mathbf{M}})$  of  $\underline{\mathbf{M}}$  is always such an example.

**8.2. Theorem.** *Let  $D: \mathcal{D} \rightarrow \mathcal{X}$  and  $E: \mathcal{X} \rightarrow \mathcal{D}$  be the hom-functors induced by the alter ego  $\underline{\mathbf{M}}$ , and let  $\mathbf{X} \in \mathcal{X}$ . Then  $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow DE(\mathbf{X})$  is an isomorphism if and only if  $\mathbf{P}_{\mathbf{X}}$  is a Priestley space. In particular,  $\underline{\mathbf{M}}$  fully dualises  $\underline{\mathbf{M}}$  at the finite level.*

*Proof.* Firstly, let  $\mathbf{X} \in \mathcal{X}$ , and assume  $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow DE(\mathbf{X})$  is an isomorphism. Thus, to prove that  $\mathbf{P}_{\mathbf{X}}$  is a Priestley space, it suffices to show that  $\mathbf{P}_{DE(\mathbf{X})}$  is a Priestley space. We will consider  $\mathbf{P}_{DE(\mathbf{X})}$  and the Priestley space  $HE(\mathbf{X})$  and prove that, for any  $u, v \in P_{DE(\mathbf{X})}$ , we have  $u \preceq v$  if and only if  $u \leq v$  in  $HE(\mathbf{X})$ . It should be noted that, as topological spaces,  $\mathbf{P}_{DE(\mathbf{X})}$  and  $HE(\mathbf{X})$  are equal.

Let  $u, v \in P_{DE(\mathbf{X})}$  with  $u \leq v$  in  $HE(\mathbf{X})$ . We want to show that there is a  $\lambda \in DE(\mathbf{X})$  such that  $\lambda$  witnesses  $u \preceq v$  at some  $1 \leq i \leq n$ . Define  $\lambda: E(\mathbf{X}) \rightarrow \underline{\mathbf{M}}$  as follows: choose any  $i, j$  with  $1 \leq i \leq \nu$  and  $2 \leq j \leq |C_i|$ , and then, for all  $a \in E(\mathbf{X})$ , define

$$\lambda(a) := \begin{cases} 1 & \text{if } u(a) = v(a) = 1, \\ \uparrow f_j^i & \text{if } u(a) = 0 \text{ and } v(a) = 1, \\ 0 & \text{if } u(a) = v(a) = 0. \end{cases}$$

Clearly  $\lambda \in DE(\mathbf{X})$ , and it is easily checked that  $f_1^i(\lambda) = u$  and  $f_{|C_i|}^i(\lambda) = v$ . If  $u \preceq v$ , then  $u \leq v$  by an easy pointwise argument. Thus  $\mathbf{P}_{DE(\mathbf{X})}$  is isomorphic to  $HE(\mathbf{X})$  and hence  $\mathbf{P}_{\mathbf{X}}$  is a Priestley space.

Conversely, assume  $\mathbf{P}_{\mathbf{X}}$  is a Priestley space. We shall show that the map  $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow DE(\mathbf{X})$  is an isomorphism. We first prove that the distributive lattice  $E(\mathbf{X})$ , with underlying set  $\mathcal{X}(\mathbf{X}, \underline{\mathbf{M}})$ , is isomorphic to  $K(\mathbf{P}_{\mathbf{X}})$ , with underlying set  $\mathcal{P}(\mathbf{P}_{\mathbf{X}}, \underline{\mathbf{D}})$ . For each  $\alpha \in E(\mathbf{X})$ , define  $\alpha^* := \alpha \upharpoonright_{P_{\mathbf{X}}}: P_{\mathbf{X}} \rightarrow D = \{0, 1\}$ . The map  $\alpha^*$  is well defined since  $\alpha$  preserves every  $f \in H(\underline{\mathbf{M}})$ . Since  $\alpha$  is continuous, so is  $\alpha^*$ . Moreover, for all  $u, v \in P_{\mathbf{X}}$ ,

$$\begin{aligned} u \preceq v &\iff (\exists x \in X) f_1^i(x) = u \text{ and } f_{|C_i|}^i(x) = v, \text{ for some } 1 \leq i \leq n \\ &\implies \alpha^*(u) = \alpha(f_1^i(x)) = f_1^i(\alpha(x)) \text{ and } \alpha^*(v) = \alpha(f_{|C_i|}^i(x)) = f_{|C_i|}^i(\alpha(x)) \\ &\implies \alpha^*(u) \leq \alpha^*(v). \end{aligned}$$

Hence,  $\alpha^* \in \mathbf{K}(\mathbf{P}_{\mathbf{X}})$ . For each  $\varphi \in \mathbf{K}(\mathbf{P}_{\mathbf{X}})$ , define  $\varphi^*: X \rightarrow M$  by

$$\varphi^*(x) := \sigma(\varphi(f_1^1(x)), \dots, \varphi(f_{|C_n|}^n(x))),$$

for all  $x \in X$ . Now let  $\varphi \in \mathbf{K}(\mathbf{P}_{\mathbf{X}})$ . For all  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ , we have  $f_j^i(x) \preceq f_{j+1}^i(x)$ , by Lemma 6.2. Thus, as  $\varphi$  is order-preserving, we conclude that  $\varphi(f_j^i(x)) \leq \varphi(f_{j+1}^i(x))$  in  $\underline{\mathbf{D}}$ . Since  $(a_1^1, \dots, a_{|C_n|}^n) \in \text{dom}(\sigma)$  if and only if  $a_j^i \leq a_{j+1}^i$  in  $\underline{\mathbf{D}}$  for all  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ , it follows that  $\varphi^*$  is well defined. We now show that  $\varphi^* \in \mathbf{E}(\mathbf{X})$ . Since  $\varphi$  and each  $f \in \mathbf{H}(\underline{\mathbf{M}})$  is continuous, so is  $\varphi^*$ . For all  $1 \leq i \leq n$ , all  $1 \leq j \leq |C_i|$  and all  $x \in X$ ,

$$\begin{aligned} \varphi^*(f_j^i(x)) &= \sigma(\varphi(f_1^1(f_j^i(x))), \dots, \varphi(f_{|C_n|}^n(f_j^i(x)))) && \text{definition of } \varphi^* \\ &= \sigma(\varphi(f_j^i(x)), \dots, \varphi(f_j^i(x))) && \text{as } \mathbf{X} \models (1) \\ &= \sigma(f_1^1(\varphi(f_j^i(x))), \dots, f_{|C_n|}^n(\varphi(f_j^i(x)))) && \text{as } \varphi(f_j^i(x)) \in \{0, 1\} \\ &= \varphi(f_j^i(x)) && \text{definition of } \sigma \\ &= f_j^i(\sigma(\varphi(f_1^1(x)), \dots, \varphi(f_{|C_n|}^n(x)))) && \text{by Lemma 7.1(a)} \\ &= f_j^i(\varphi^*(x)) && \text{definition of } \varphi^*. \end{aligned}$$

So  $\varphi^*$  preserves every  $f \in \mathbf{H}(\underline{\mathbf{M}})$ . For all  $1 \leq i \leq \nu$ , all  $1 \leq j \leq |C_i| - 1$  and  $x \in X$ ,

$$\begin{aligned} g_j^i(\varphi^*(x)) &= g_j^i(\sigma(\varphi(f_1^1(x)), \dots, \varphi(f_{|C_n|}^n(x)))) && \text{definition of } \varphi^* \\ &= \sigma(b_1^1, \dots, b_{|C_n|}^n) && \text{by Lemma 7.1(c),} \\ & \quad \text{where } b_{j'}^{i'} = \begin{cases} \varphi(f_{j+1}^{i'}(x)), & \text{if } f_{j'}^{i'} \in \uparrow f_{j+1}^{i'}; \\ \varphi(f_j^{i'}(x)), & \text{if } f_{j'}^{i'} \notin \uparrow f_{j+1}^{i'}. \end{cases} \\ &= \sigma(\varphi(f_1^1(g_j^i(x))), \dots, \varphi(f_{|C_n|}^n(g_j^i(x)))) && \text{by (4)} \\ &= \varphi^*(g_j^i(x)) && \text{definition of } \varphi^*. \end{aligned}$$

Thus  $\varphi^*$  preserves each  $g \in G$ .

Let  $(x_1^1, \dots, x_{|C_{\nu|-1}^{\nu}}, x_1^n) \in \text{dom}(q)$ . As  $\varphi^*$  preserves every  $f \in \mathbf{H}(\underline{\mathbf{M}})$ , it follows from (6) that  $(\varphi^*(x_1^1), \dots, \varphi^*(x_{|C_{\nu|-1}^{\nu}}), \dots, \varphi^*(x_1^n)) \in \text{dom}(q)$ . Now,

$$\begin{aligned} & q(\varphi^*(x_1^1), \dots, \varphi^*(x_{|C_{\nu|-1}^{\nu}}), \dots, \varphi^*(x_1^n)) \\ &= \sigma(f_1^1(\varphi^*(x_1^1)), \dots, f_1^1(\varphi^*(x_{|C_1|-1}^1)), f_{|C_1|}^1(\varphi^*(x_{|C_1|-1}^1)), \\ & \quad \dots, f_1^{\nu}(\varphi^*(x_1^{\nu})), \dots, f_1^{\nu}(\varphi^*(x_{|C_{\nu|-1}^{\nu}})), f_{|C_{\nu}|}^{\nu}(\varphi^*(x_{|C_{\nu|-1}^{\nu}})), \\ & \quad f_1^{\nu+1}(\varphi^*(x_1^{\nu+1})), \dots, f_1^n(\varphi^*(x_1^n))) && \text{by Lemma 7.1(b)} \\ &= \sigma(\varphi(f_1^1(x_1^1)), \dots, \varphi(f_1^1(x_{|C_1|-1}^1)), \varphi(f_{|C_1|}^1(x_{|C_1|-1}^1)), \\ & \quad \dots, \varphi(f_1^{\nu}(x_1^{\nu})), \dots, \varphi(f_1^{\nu}(x_{|C_{\nu|-1}^{\nu}})), \varphi(f_{|C_{\nu}|}^{\nu}(x_{|C_{\nu|-1}^{\nu}})), \\ & \quad \varphi(f_1^{\nu+1}(x_1^{\nu+1})), \dots, \varphi(f_1^n(x_1^n))) && \text{definition of } \varphi^* \text{ then Lemma 7.1(a)} \end{aligned}$$

$$\begin{aligned}
&= \sigma(\varphi(f_1^1(q(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n))), \\
&\quad \dots, \varphi(f_{|C_\nu|}^\nu(q(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n))), \\
&\quad \varphi(f_1^{\nu+1}(q(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n))), \\
&\quad \dots, \varphi(f_1^n(q(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n)))) \quad \text{as } \mathbf{X} \models (7) \\
&= \varphi^*(q(x_1^1, \dots, x_{|C_\nu|-1}^\nu, \dots, x_1^n)) \quad \text{definition of } \varphi^*.
\end{aligned}$$

Thus  $\varphi^*$  preserves  $q$ . Let  $1 \leq i, i' \leq \nu$  and  $(x, y) \in \text{dom}(s_i^{i'})$ . As  $\varphi^*$  preserves both  $f_{|C_i|}^i$  and  $f_1^{i'}$ , we have  $(\varphi^*(x), \varphi^*(y)) \in \text{dom}(s_i^{i'})$ , so

$$\begin{aligned}
s_i^{i'}(\varphi^*(x), \varphi^*(y)) &= \sigma(\underbrace{f_1^i(\varphi^*(x)), \dots, f_1^i(\varphi^*(x))}_{|C_1|-1}, \underbrace{f_{|C_{i'}|}^{i'}(\varphi^*(y))}_{n-\nu}, \\
&\quad \dots, \underbrace{f_1^i(\varphi^*(x)), \dots, f_1^i(\varphi^*(x))}_{|C_\nu|-1}, \underbrace{f_{|C_{i'}|}^{i'}(\varphi^*(y))}_{n-\nu}), \quad \text{by Lemma 7.1(d)} \\
&= \sigma(\underbrace{\varphi(f_1^i(x)), \dots, \varphi(f_1^i(x))}_{|C_1|-1}, \varphi(f_{|C_{i'}|}^{i'}(y)), \\
&\quad \dots, \underbrace{\varphi(f_1^i(x)), \dots, \varphi(f_1^i(x))}_{|C_\nu|-1}, \varphi(f_{|C_{i'}|}^{i'}(y)), \\
&\quad \underbrace{\varphi(f_{|C_{i'}|}^{i'}(y)), \dots, \varphi(f_{|C_{i'}|}^{i'}(y))}_{n-\nu}) \quad \begin{array}{l} \text{definition of } \varphi^*, \\ \text{then Lemma 7.1(a)} \end{array} \\
&= \sigma(\varphi(f_1^1(s_i^{i'}(x, y))), \dots, \varphi(f_{|C_n|}^n(s_i^{i'}(x, y)))) \quad \text{as } \mathbf{X} \models (9) \\
&= \varphi^*(s_i^{i'}(x, y)) \quad \text{definition of } \varphi^*.
\end{aligned}$$

Thus  $\varphi^*$  preserves all  $s \in S$ , and so we have proved that  $\varphi^* \in \mathbf{E}(\mathbf{X})$ .

If  $\alpha \leq \beta$  in  $\mathbf{E}(\mathbf{X})$ , then clearly  $\alpha^* \leq \beta^*$  in  $\mathbf{K}(\mathbf{P}_\mathbf{X})$ . If  $\varphi \leq \psi$  in  $\mathbf{K}(\mathbf{P}_\mathbf{X})$ , then for all  $x \in X$  we have  $\varphi(f(x)) \leq \psi(f(x))$ , for each  $f \in \mathbf{H}(\mathbf{M})$ . So, as  $\sigma$  is order-preserving,

$$\varphi^*(x) = \sigma(\varphi(f_1^1(x)), \dots, \varphi(f_{|C_n|}^n(x))) \leq \sigma(\psi(f_1^1(x)), \dots, \psi(f_{|C_n|}^n(x))) = \psi^*(x).$$

Hence,  $\varphi^* \leq \psi^*$  in  $\mathbf{E}(\mathbf{X})$ . For all  $x \in X$  and  $\alpha \in \mathbf{E}(\mathbf{X})$ , we have

$$\begin{aligned}
\alpha^{**}(x) &= (\alpha \upharpoonright_{P_X})^*(x) \\
&= \sigma(\alpha \upharpoonright_{P_X}(f_1^1(x)), \dots, \alpha \upharpoonright_{P_X}(f_{|C_n|}^n(x))) \\
&= \sigma(\alpha(f_1^1(x)), \dots, \alpha(f_{|C_n|}^n(x))) \\
&= \sigma(f_1^1(\alpha(x)), \dots, f_{|C_n|}^n(\alpha(x))) \\
&= \alpha(x) \quad \text{definition of } \sigma.
\end{aligned}$$

Thus  $\alpha^{**} = \alpha$ . For all  $u \in P_X$  and  $\varphi \in K(\mathbf{P}_X)$ , we have

$$\begin{aligned}
\varphi^{**}(u) &= \varphi^* \upharpoonright_{P_X}(u) \\
&= \varphi^*(u) \\
&= \sigma(\varphi(f_1^1(u)), \dots, \varphi(f_{|C_n|}^n(u))) \\
&= \sigma(\varphi(u), \dots, \varphi(u)) && \text{as } u \in D \\
&= \sigma(f_1^1(\varphi(u)), \dots, f_{|C_n|}^n(\varphi(u))) && \text{as } \varphi(u) \in D \\
&= \varphi(u) && \text{definition of } \sigma,
\end{aligned}$$

whence  $\varphi^{**} = \varphi$ . We have proved that the maps  $\Phi: \alpha \mapsto \alpha^*$  and  $\Psi: \varphi \mapsto \varphi^*$  are mutually inverse lattice isomorphisms.

In order to prove that  $\varepsilon_{\mathbf{X}}: \mathbf{X} \rightarrow \mathbf{DE}(\mathbf{X})$  is surjective, it remains to show that if  $k: E(\mathbf{X}) \rightarrow \underline{\mathbf{M}}$  is a 0, 1-lattice homomorphism, then there exists  $x \in X$  such that  $k = \varepsilon_{\mathbf{X}}(x)$ , that is, that  $k(\alpha) = \alpha(x)$ , for all  $\alpha \in E(\mathbf{X})$ . Define  $l := k \circ \Psi: K(\mathbf{P}_X) \rightarrow \underline{\mathbf{M}}$ , then  $f_1^i \circ l \leq \dots \leq f_{|C_i|}^i \circ l$  in  $\mathbf{HK}(\mathbf{P}_X)$ , for all  $1 \leq i \leq n$ . Since Priestley duality is full, there exist  $u_1^i, \dots, u_{|C_i|}^i \in P_X$  such that  $u_1^i \preceq \dots \preceq u_{|C_i|}^i$  and  $(f_1^i \circ l)(\varphi) = \varphi(u_1^i), \dots, (f_{|C_i|}^i \circ l)(\varphi) = \varphi(u_{|C_i|}^i)$ , for all  $\varphi \in K(\mathbf{P}_X)$ . Since  $u_j^i \preceq u_{j+1}^i$ , for all  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ , there exists an  $x_j^{i'} \in X$  that witnesses  $u_j^i \preceq u_{j+1}^i$  at  $i'$ , for some  $1 \leq i' \leq \nu$ . Define  $x_j^i := s_i^{i'}(u_j^i, x_j^{i'})$ , for all  $1 \leq i \leq \nu$  and  $1 \leq j \leq |C_i| - 1$ . By (9) and (1) it follows that  $x_j^i$  witnesses  $u_j^i \preceq u_{j+1}^i$  at  $i$  and therefore

$$(x_1^1, \dots, x_{|C_1|-1}^1, \dots, x_1^\nu, \dots, x_{|C_\nu|-1}^\nu, u_1^{\nu+1}, \dots, u_1^n) \in \text{dom}(q).$$

Define  $x := q(x_1^1, \dots, x_{|C_1|-1}^1, \dots, x_1^\nu, \dots, x_{|C_\nu|-1}^\nu, u_1^{\nu+1}, \dots, u_1^n)$ . We will show that  $k(\alpha) = \alpha(x)$ , for all  $\alpha \in E(\mathbf{X})$ . For all  $\alpha \in E(\mathbf{X})$ , all  $1 \leq i \leq n$  and all  $1 \leq j \leq |C_i|$ , we have

$$\begin{aligned}
f_j^i(l(\alpha^*)) &= \alpha^*(u_j^i) \implies f_j^i(k(\Psi(\Phi(\alpha)))) = \alpha^*(u_j^i) \\
&\implies f_j^i(k(\alpha)) = \alpha(u_j^i) \\
&\implies f_j^i(k(\alpha)) = \alpha(f_j^i(x)) && \text{as } \mathbf{X} \models (7) \\
&\implies f_j^i(k(\alpha)) = f_j^i(\alpha(x)),
\end{aligned}$$

whence  $k(\alpha) = \alpha(x)$ , since the elements of  $\mathbf{H}(\underline{\mathbf{M}})$  separate points of  $M$ .  $\square$

As mentioned earlier, it is proved in [6] that every full duality for  $\mathcal{D}$  based on a finite non-boolean bounded distributive lattice is actually strong. Consequently, by Lemma 8.1,  $\underline{\mathbf{M}}$  does not yield a full duality on  $\mathcal{D}$ . Thus, there must exist an infinite  $\mathbf{X} \in \mathcal{X}$  for which  $\mathbf{P}_X$  is not a Priestley space. An explicit example of such a structure is known only in the case that  $\underline{\mathbf{M}}$  is the three-element chain (see [8]).

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