

Near unanimity: an obstacle to general duality theory

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Until now, papers on duality theory within general algebra have concentrated on finding conditions on a finite non-trivial algebra \mathbf{P} which will allow us to prove easily that there is a natural duality for the quasivariety $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{P})$ generated by \mathbf{P} (see [9; 10; 11; 4]) or, more recently, on ways to refine an existing duality by deleting unnecessary structure from the objects in the dual category (see [5; 6]). This paper begins the attack on a much more fundamental question: “Which finite algebras \mathbf{P} admit a natural duality?” (We say that \mathbf{P} admits a natural duality or, when we are being more colloquial, is *dualizable*, if there is a natural duality for the quasivariety generated by \mathbf{P} .)

Davey and Werner [9] showed that if \mathbf{P} has a *near-unanimity term*, that is, a term t of arity at least three such that \mathbf{P} satisfies the identities

$$t(x, \dots, x, y) \approx t(x, \dots, x, y, x) \approx \dots \approx t(y, x, \dots, x) \approx x,$$

then \mathbf{P} admits a natural duality. Thus, near-unanimity yields a host of dualizable algebras. Nevertheless, we shall show that it is a two-edged sword. Near-unanimity is actually an obstacle to the creation of examples of dualizable algebras: in the presence of even a small degree of congruence distributivity the only dualizable algebras are those which possess a near-unanimity term. To some extent this explains the difficulty encountered when attempting to find algebras which admit a duality but do not possess a near-unanimity term.

We show, for example, that if \mathbf{P} is a finite algebra which admits a natural duality and generates a congruence distributive (or even congruence join-semidistributive) variety, then \mathbf{P} has a near-unanimity term. We show also that if \mathbf{P} is a finite order-primal algebra which admits a natural duality, then again \mathbf{P} must have a near-unanimity term. En route we obtain a topological characterization (independent of duality theory) of finite algebras which possess a near-unanimity term.

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In the remainder of this introduction we briefly review the general framework of natural dualities. Then Section 1 is devoted to the statement and discussion of the results. Some observations on duality in general are given in Section 2. The proofs of the main theorems are given in the final section.

We shall follow the notation of Davey [3], which gives an overview of duality theory. (Some of the results of this paper were announced without proof in [3].) Denote the discrete topology on P by \mathcal{T} . Let $\mathbf{P} = \langle P; G, H, R, \mathcal{T} \rangle$ be a structure on the same base set P as the algebra $\mathbf{P} = \langle P; F \rangle$ and assume that $G \cup H \cup R$ is algebraic over \mathbf{P} , that is, each n -ary operation $g \in G$ is a homomorphism $g : \mathbf{P}^n \rightarrow \mathbf{P}$, for each n -ary partial operation $h \in H$ its domain A is a subuniverse of \mathbf{P}^n and $h : A \rightarrow \mathbf{P}$ is a homomorphism, and each n -ary relation $r \in R$ is a subuniverse of \mathbf{P}^n . Define $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}(\mathbf{P})$ to be the class of all isomorphic copies of (topologically) closed substructures of powers of \mathbf{P} . We then have well-defined hom-functors $D : \mathcal{A} \rightarrow \mathcal{X}$ and $E : \mathcal{X} \rightarrow \mathcal{A}$ defined on objects by

$$D(\mathbf{A}) := \mathcal{A}(\mathbf{A}, \mathbf{P}) \leq \mathbf{P}^A \quad \text{and} \quad E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \mathbf{P}) \leq \mathbf{P}^X$$

for all $\mathbf{A} \in \mathcal{A}$ and all $\mathbf{X} \in \mathcal{X}$. For each $\mathbf{A} \in \mathcal{A}$ the evaluation map

$$e_{\mathbf{A}} : \mathbf{A} \rightarrow ED(\mathbf{A}) = \mathcal{X}(\mathcal{A}(\mathbf{A}, \mathbf{P}), \mathbf{P}),$$

given by

$$(e_{\mathbf{A}}(a))(x) := x(a) \quad \text{for all } x \in D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{P})$$

is an embedding. If $e_{\mathbf{A}}$ is an isomorphism, then we say that \mathbf{P} yields a natural duality on \mathbf{A} . When \mathbf{P} yields a natural duality on all $\mathbf{A} \in \mathcal{A}$, we say that \mathbf{P} yields a natural duality on \mathcal{A} . Thus the algebra \mathbf{P} admits a natural duality (or is dualizable) provided there is some structure \mathbf{P} which yields a natural duality on the quasivariety $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{P})$ generated by \mathbf{P} .

For a detailed discussion of the theory of natural dualities and references to the literature we refer the reader to Davey [3].

1. Duality and near unanimity *

The NU-Duality Theorem (see Davey [3, Theorem 2.8] or Davey and Werner [9, Theorem 1.19]) tells us that if \mathbf{P} has a 3-ary near-unanimity term (better known as a majority term), then \mathbf{P} admits a particularly well-behaved duality since \mathbf{P} can

be chosen such that

- (a) \mathbf{P} is of finite type (that is, $G \cup H \cup R$ is finite),
- (b) the operations in G and the partial operations in H are at most unary and the relations in R are at most binary, and
- (c) \mathbf{P} is injective in \mathcal{X} .

Our first theorem provides a partial converse. The full subcategory of \mathcal{X} whose objects are the finite members of \mathcal{X} is denoted by \mathcal{X}_{fin} .

1.1 THEOREM. *Let \mathbf{P} be a finite non-trivial algebra. Assume that \mathbf{P} is of finite type and that the (partial) operations in $G \cup H$ are at most unary. Let m be the maximum of the arities of the relations in R . If \mathbf{P} yields a duality on \mathcal{A} with \mathbf{P} injective in \mathcal{X}_{fin} , then \mathbf{P} has a $(k + 1)$ -ary near-unanimity term where $k = \max\{2, m\}$.*

Proof. Let \mathbf{P} and k be as described in the statement of the theorem. For $a, b \in P$, define

$$X_{ab} := \{(a, \dots, a, b), (a, \dots, b, a), \dots, (b, a, \dots, a)\} \subseteq P^{k+1}$$

and define $X := \bigcup \{X_{ab} \mid a, b \in P\}$. Since the (partial) operations in $G \cup H$ are at most unary, X is a substructure of P^{k+1} . Define $\alpha : X \rightarrow P$ by $\alpha(x) = a$ if $x \in X_{ab}$. On any k -or-fewer-element subset of X , the map α is a projection. Hence α preserves all the relations in R since they are at most k -ary, and clearly α preserves the (partial) operations in $G \cup H$ as they are at most unary. Thus $\alpha \in \mathcal{X}(X, \mathbf{P})$. Since \mathbf{P} is injective in \mathcal{X}_{fin} , there is a morphism $t : P^{k+1} \rightarrow \mathbf{P}$ which extends α . As \mathbf{P} yields a duality on \mathcal{A} , the map t is a $(k + 1)$ -ary term function on \mathbf{P} (see [3, Theorem 2.5] or Davey and Werner [9, Theorem 1.8]) and $t \upharpoonright X = \alpha$ says exactly that t is a near-unanimity function. □

Whereas this result shows that the nature of the structure \mathbf{P} can force \mathbf{P} to have a near-unanimity term, our main theorem provides simple algebraic conditions on the algebra \mathbf{P} which in the presence of dualizability force \mathbf{P} to have a near-unanimity term.

Recall that a lattice is *join-semi-distributive* if it satisfies the quasi-identity

$$x \vee y = x \vee z \quad \Rightarrow \quad x \vee (y \wedge z) = x \vee y.$$

Clearly distributive lattices are join-semi-distributive. Associated with an algebra $\mathbf{A} \in \mathcal{A}$ are two natural lattices of congruences. The first is simply the lattice $\text{Con}(\mathbf{A})$

of all congruences. The second is the lattice $\text{Con}_{\mathcal{A}}(\mathbf{A})$ consisting of $V = A^2$ along with all \mathcal{A} -congruences (also known as *relative congruences*), that is, those congruences θ on \mathbf{A} such that $\mathbf{A}/\theta \in \mathcal{A}$. Since \mathcal{A} need not be a variety (and therefore closed under homomorphic images), $\text{Con}_{\mathcal{A}}(\mathbf{A})$ is, in general, a proper subset of $\text{Con}(\mathbf{A})$.

Note that $\text{Con}_{\mathcal{A}}(\mathbf{A})$ is closed under arbitrary meets in $\text{Con}(\mathbf{A})$ but need not be a sublattice. An algebra $\mathbf{A} \in \mathcal{A}$ is called *congruence distributive* if $\text{Con}(\mathbf{A})$ is distributive and is called *relatively congruence distributive* if $\text{Con}_{\mathcal{A}}(\mathbf{A})$ is distributive. *Congruence join-semi-distributivity* and *relative congruence join-semi-distributivity* are defined similarly. It is well known and if $\underline{\mathbf{P}}$ has a near-unanimity term, then every algebra in the variety generated by $\underline{\mathbf{P}}$, and hence every algebra in $\mathbb{ISP}(\underline{\mathbf{P}})$, is congruence distributive. In general, it is not clear how properties of $\text{Con}(\mathbf{A})$ and $\text{Con}_{\mathcal{A}}(\mathbf{A})$ are related. We do not even know whether the existence of a near-unanimity term implies (some weak form of) distributivity of $\text{Con}_{\mathcal{A}}(\mathbf{A})$.

1.2 THE NU-OBSTACLE THEOREM. *Let $\underline{\mathbf{P}}$ be a finite non-trivial algebra and let $\mathcal{A} := \mathbb{ISP}(\underline{\mathbf{P}})$. The following are equivalent:*

- (a) $\underline{\mathbf{P}}$ has a near-unanimity term;
- (b) $\underline{\mathbf{P}}$ generates a congruence-distributive variety and $\underline{\mathbf{P}}$ admits a duality;
- (c) every finite algebra in \mathcal{A} is congruence join-semi-distributive and $\underline{\mathbf{P}}$ admits a duality.

Furthermore, if every finite algebra in \mathcal{A} is relatively congruence join-semi-distributive and $\underline{\mathbf{P}}$ admits a duality, then $\underline{\mathbf{P}}$ has a near-unanimity term.

Since there is a large gap between congruence (join-semi-) distributivity and near unanimity, the NU-Obstacle Theorem allows us to find many examples of algebras which do not admit a natural duality. For example, let $\underline{\mathbf{P}} = \langle \{0, 1\}; \rightarrow \rangle$ be the two-element implication algebra, where $x \rightarrow y := x' \vee y$. Then, by [14], the algebra $\underline{\mathbf{P}}$ generates a congruence-distributive variety but $\underline{\mathbf{P}}$ has no near-unanimity term. Thus $\underline{\mathbf{P}}$ does not admit a duality. (This was proved directly in Section 2.1 on pages 148–151 of [9].) There is a countable number of non-term-equivalent algebras on the set $\{0, 1\}$. The two-element implication algebra is one of exactly eight which are congruence distributive but have no near-unanimity term and so do not admit a duality. All of the others admit a duality (Davey and Rosenberg [8]) – see Section 7 of Davey [3].

The second part of the NU-Obstacle Theorem has an immediate application to order-primal algebras. An algebra $\underline{\mathbf{P}} = \langle P; F \rangle$ is *order-primal* if there is some order on P such that, for all $n \in \mathbb{N}$, a map $t : P^n \rightarrow P$ is a term function on $\underline{\mathbf{P}}$ just when t preserves this order. For example, the two-element bounded distributive lattice is order-primal with respect to its underlying order. It is proved in Davey, Quacken-

bush and Schweigert [7] (see also McKenzie [13]) that if \mathbf{P} is order-primal, then $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{P})$ is relatively congruence distributive. Hence, the theorem below, which improves Theorem 2.3 of [7], is an immediate corollary of the NU–Obstacle Theorem.

1.3 THEOREM. *A finite order-primal algebra \mathbf{P} admits a duality if and only if \mathbf{P} has a near-unanimity term.*

It is now very easy to find order-primal algebras which do not admit a duality. For example, if $\langle P; \leq \rangle$ is a crown with more than four elements, and \mathbf{P} is an order-primal algebra with respect to $\langle P; \leq \rangle$, then \mathbf{P} has no near-unanimity term (see [2]) and hence \mathbf{P} does not admit a duality. L. Zadori [15] has found some beautiful characterizations of finite bounded ordered sets which admit a monotone near-unanimity operation, showing, in particular, that the corresponding order-primal algebra has a near-unanimity term if and only if the variety it generates is congruence distributive.

In the process of proving the NU–Obstacle Theorem, we shall prove the following, characterization of near unanimity. The connection between dualizability and the purely topological condition given in (b) below is explained in the following section. We denote the discrete topological space $(P; \mathcal{F})$ by \mathbf{P} , and, for each topological space \mathbf{X} , we denote by $C(\mathbf{X}, \mathbf{P})$ the subalgebra of $\mathbf{P}^{\mathbf{X}}$ consisting of the continuous maps from \mathbf{X} into \mathbf{P} . As usual, we endow $\mathbf{P}^{\mathbf{X}}$ with the product topology. We refer to the induced subspace topology on $C(\mathbf{X}, \mathbf{P})$ as the *pointwise topology*.

1.4 THEOREM. *Let \mathbf{P} be a finite non-trivial algebra. The following are equivalent:*

- (a) \mathbf{P} has a near-unanimity term;
- (b) \mathbf{P} generates a congruence-distributive variety and, for every compact (totally disconnected) topological space \mathbf{X} , each subalgebra of $C(\mathbf{X}, \mathbf{P})$ is closed in the pointwise topology.

2. Duality and closure in the pointwise topology

In this section, we present some useful observations on natural dualities in general. Recall that when considering whether a structure $\mathbf{P} = \langle P; G, H, R, \mathcal{F} \rangle$ yields a duality on an algebra $A \in \mathcal{A}$ we may delete some or all of the (partial) operations from $G \cup H$ provided we add their graphs to the set R of relations (see [3, Lemma 2.1]). Thus, for the remainder of this paper we shall assume that \mathbf{P} is a relational structure, $\mathbf{P} = \langle P; R, \mathcal{F} \rangle$.

If some structure yields a natural duality on \mathbf{A} , then so does the structure

$$\mathbf{P} = \langle P; B, \mathcal{T} \rangle \quad \text{where } B = \bigcup \{ \mathbb{S}(\mathbf{P}^n) \mid n \geq 1 \}.$$

This is known as the *brute force* construction. Note that as we vary the structure \mathbf{P} , the set $D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{P})$ does not vary, nor does the topology induced on $D(\mathbf{A})$ from \mathbf{P}^A . Thus the subalgebra $C(D(\mathbf{A}), \mathbf{P})$ of $\mathbf{P}^{D(\mathbf{A})}$ is independent of the choice of the structure \mathbf{P} . Let

$$A^\# := \{ e_A(a) : D(\mathbf{A}) \rightarrow \mathbf{P} \mid a \in A \}$$

be the natural isomorphic copy of \mathbf{A} within $C(D(\mathbf{A}), \mathbf{P})$. As we vary \mathbf{P} the second dual, $ED(\mathbf{A})$ of \mathbf{A} varies between $A^\#$ (when \mathbf{P} yields a duality on \mathbf{A}) and $C(D(\mathbf{A}), \mathbf{P})$ (when we choose $R = \emptyset$).

2.1 LEMMA. *For all choices of \mathbf{P} , the second dual, $ED(\mathbf{A})$, of \mathbf{A} contains $A^\#$ and is closed in the pointwise topology on $C(D(\mathbf{A}), \mathbf{P})$. Under the brute-force construction (that is, when $\mathbf{P} = \langle P; B, \mathcal{T} \rangle$), the second dual, $ED(\mathbf{A})$ of \mathbf{A} is the closure of $A^\#$ in the pointwise topology on $C(D(\mathbf{A}), \mathbf{P})$.*

Proof. First note that for $U \subseteq V \subseteq P^X$, we have U closed in V if and only if every function $v \in V$, which on each finite subset F of X agrees with some $u \in U$, is itself in U . Let $\mathbf{P} = \langle P; R, \mathcal{T} \rangle$. Since each relation in R is finitary, it follows easily that, for all $\mathbf{X} \in \mathcal{X}$ the set $E(\mathbf{X}) := \mathcal{X}(\mathbf{X}, \mathbf{P})$ is closed in $C(\mathbf{X}, \mathbf{P})$.

Next note that for $U \subseteq V \subseteq P^X$, the function v is in the closure of U in V if and only if $v \in V$ and for every finite subset F of X there exists $u \in U$ such that $v \upharpoonright F = u \upharpoonright F$. Now let $\mathbf{P} = \langle P; B, \mathcal{T} \rangle$. Since $ED(\mathbf{A})$ is closed in $C(D(\mathbf{A}), \mathbf{P})$, it remains to prove that if $\varphi \in ED(\mathbf{A})$ then φ is in the closure of $A^\#$ in $C(D(\mathbf{A}), \mathbf{P})$. Suppose, by way of contradiction, that there exists a finite subset F of $D(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{P})$ such that for all $a \in A$ we have $\varphi \upharpoonright F \neq e_A(a) \upharpoonright F$. Assume that $F = \{x_1, \dots, x_n\}$ and define an n -ary relation r on P by

$$r = \{ (x_1(a), \dots, x_n(a)) \mid a \in A \}.$$

Since x_1, \dots, x_n are homomorphisms from \mathbf{A} to \mathbf{P} , the set r is a subuniverse of \mathbf{P}^n . Thus $r \in B$. By construction, we have $(x_1, \dots, x_n) \in r_{D(\mathbf{A})}$ and hence, since φ preserves r , it follows that $(\varphi(x_1), \dots, \varphi(x_n)) \in r$. Hence there exists $a \in A$ such that $\varphi(x_i) = x_i(a)$ for $i = 1, \dots, n$, that is, $\varphi \upharpoonright F = e_A(a) \upharpoonright F$, a contradiction. \square

Following Heindorf [12], for a compact space \mathbf{X} we say that a subalgebra \mathbf{B} of $C(\mathbf{X}, \mathbf{P})$ is *balanced* if B separates the points of \mathbf{X} (that is, if $x, y \in X$ and for all

$b \in B$ we have $b(x) = b(y)$, then $x = y$) and every homomorphism $u : \mathbf{B} \rightarrow \mathbf{P}$ is given by evaluation at some point $x \in X$ (that is, $u = \pi_x \upharpoonright B$ where $\pi_x : P^X \rightarrow P$ is the natural projection). The interest in balanced algebras stems from the following theorem and the two lemmas which precede it.

2.2 LEMMA. *For all $\mathbf{A} \in \mathcal{A}$, the algebra $\mathbf{A}^\#$ is a balanced subalgebra of $C(D(\mathbf{A}), \mathbf{P})$ and hence every algebra in \mathcal{A} is isomorphic to a balanced subalgebra of $C(\mathbf{X}, \mathbf{P})$ for some compact space \mathbf{X} .*

Proof. It is trivial that $\mathbf{A}^\#$ separates the points of $D(\mathbf{A})$. Let $u : \mathbf{A}^\# \rightarrow \mathbf{P}$ be a homomorphism and define $x := u \circ e_{\mathbf{A}}$. Then $x \in D(\mathbf{A})$ and, for all $a \in A$,

$$u(e_{\mathbf{A}}(a)) = (u \circ e_{\mathbf{A}})(a) = x(a) = (e_{\mathbf{A}}(a))(x) = \pi_x(e_{\mathbf{A}}(a)),$$

that is $u = \pi_x \upharpoonright \mathbf{A}^\#$. Hence $\mathbf{A}^\#$ is a balanced subalgebra of $C(D(\mathbf{A}), \mathbf{P})$. \square

We leave the proof of our next lemma as an easy exercise.

2.3 LEMMA. *Let \mathbf{X} be a compact space and let \mathbf{B} be a subalgebra of $C(D(\mathbf{A}), \mathbf{P})$. Then \mathbf{B} is balanced precisely when the map $\varphi : X \rightarrow D(\mathbf{B})$, defined by $\varphi(x) := \pi_x \upharpoonright B$, is a homeomorphism.*

Note that it follows from this lemma that if \mathbf{X} is a compact space and \mathbf{B} is a balanced subalgebra of $C(\mathbf{X}, \mathbf{P})$, then \mathbf{X} is in fact a Boolean space.

2.4 THEOREM. *Let \mathbf{P} be a finite non-trivial algebra and let $\mathbf{A} \in \mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{P})$. The following are equivalent:*

- (a) *there is some structure \mathbf{P} which yields a duality on \mathbf{A} ;*
- (b) *brute force yields a duality on \mathbf{A} ;*
- (c) *$\mathbf{A}^\# := \{e_{\mathbf{A}}(a) : D(\mathbf{A}) \rightarrow \mathbf{P} \mid a \in A\}$ is closed in the pointwise topology on $C(D(\mathbf{A}), \mathbf{P})$;*
- (d) *if \mathbf{A} is isomorphic to a balanced subalgebra \mathbf{B} of $C(\mathbf{X}, \mathbf{P})$ for some compact space \mathbf{X} , then \mathbf{B} is closed in $C(\mathbf{X}, \mathbf{P})$;*
- (e) *\mathbf{A} is isomorphic to a closed and balanced subalgebra of $C(\mathbf{X}, \mathbf{P})$ for some compact space \mathbf{X} .*

Proof. The equivalence of (a), (b) and (c) follows from Lemma 2.1 while Lemma 2.2 shows that (c) implies (e) and that (d) implies (c). Thus it remains to prove that (e) implies (d).

Assume that \mathbf{B} is a balanced subalgebra of $C(\mathbf{X}, \mathbf{P})$, that \mathbf{C} is a balanced subalgebra of $C(\mathbf{Y}, \mathbf{P})$ and that $u : \mathbf{C} \rightarrow \mathbf{B}$ is an isomorphism. We claim that if C is closed in $C(\mathbf{Y}, \mathbf{P})$, then B is closed in $C(\mathbf{X}, \mathbf{P})$. That (e) implies (d) then follows easily. Let $D(u) : D(\mathbf{B}) \rightarrow D(\mathbf{C})$ be the homeomorphism induced by the isomorphism u , let $\varphi_X : X \rightarrow D(\mathbf{B})$ and $\varphi_Y : Y \rightarrow D(\mathbf{C})$ be the homeomorphisms given by Lemma 2.3 and let $\psi : X \rightarrow Y$ be a homeomorphism which satisfies $D(u) \circ \varphi_X = \varphi_Y \circ \psi$. The map ψ induces a homeomorphism $\Psi : C(\mathbf{Y}, \mathbf{P}) \rightarrow C(\mathbf{X}, \mathbf{P})$. A simple diagram chase shows that Ψ satisfies $\Psi(c) = u(c)$ for all $c \in C$ from which we conclude that $\Psi(C) = B$. Consequently, if C is closed in $C(\mathbf{Y}, \mathbf{P})$, then B is closed in $C(\mathbf{X}, \mathbf{P})$. \square

Thus, to prove that \mathbf{P} does not admit a duality it suffices to find a Boolean space \mathbf{X} and a balanced subalgebra \mathbf{B} of $C(\mathbf{X}, \mathbf{P})$ which is not closed in the pointwise topology. This has been used in Heindorf [12] to show that for $n \geq 3$ there are 2^ω non-term-equivalent non-dualizable algebras on the set $\{0, 1, \dots, n - 1\}$ and to prove that a pre-primal algebra admits a duality if and only if it is linear or has a near-unanimity term.

3. The proofs of Theorems 1.2 and 1.4

Theorem 1.2 will follow from the following slightly more general result. The proof of the Con-version is converted into a proof of the $\text{Con}_{\mathcal{A}}$ -version by the occasional replacement of the word “congruence” by “ \mathcal{A} -congruence” [which we indicate in brackets].

We say that a subalgebra \mathbf{B} of $C(\mathbf{X}, \mathbf{P})$ is *well placed* if every homomorphism $u : \mathbf{B} \rightarrow \mathbf{P}$ is continuous with respect to the pointwise topology on B and the discrete topology on P . It follows that a homomorphism from \mathbf{B} into a finite member of \mathcal{A} is continuous since each of its “coordinate” functions is continuous. Since the evaluation maps $\pi_x : P^X \rightarrow P$ are continuous for all $x \in X$, every balanced subalgebra of $C(\mathbf{X}, \mathbf{P})$ is well placed. In particular, $\mathbf{A}^\#$ is a well-placed subalgebra of $C(D(\mathbf{A}), \mathbf{P})$.

3.1 THEOREM. *Let \mathbf{P} be a finite non-trivial algebra and assume that every finite algebra in $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}(\mathbf{P})$ is [relatively] join-semi-distributive. If every algebra $\mathbf{A} \in \mathcal{A}$ is isomorphic to a closed and well-placed subalgebra of $C(\mathbf{X}, \mathbf{P})$ for some compact space \mathbf{X} , then \mathbf{P} has a near-unanimity term.*

Proof. Let \mathbf{F} denote the algebra freely generated in \mathcal{A} by two elements c and d , let \mathbf{F}^ω be the set of all infinite sequence of elements from \mathbf{F} and let \mathbf{A} be the subalgebra of \mathbf{F}^ω generated by the set of sequences $\{s_n \mid n < \omega\}$, where $s_n(m) := c$ for

$m \neq n$ while $s_n(n) := d$. By assumption, there is an isomorphism $a \mapsto \bar{a}$ mapping \mathbf{A} onto a closed and well-placed subalgebra \mathbf{B} of $\mathbf{C}(\mathbf{X}, \mathbf{P})$ for some compact space \mathbf{X} . In particular, $\{\bar{s}_n\}_{n < \omega}$ is a sequence of continuous functions from \mathbf{X} to \mathbf{P} .

Assume for the moment that we have already proved that the sequence converges pointwise to some function $b \in \mathbf{C}(\mathbf{X}, \mathbf{P})$. Then $b \in \mathbf{B}$, since \mathbf{B} is closed, and since \mathbf{B} is generated by $\{\bar{s}_n \mid n < \omega\}$ there is some term t in the language of \mathbf{P} such that

$$b = t(\bar{s}_0, \dots, \bar{s}_k) = \overline{t(s_0, \dots, s_k)}.$$

The algebra \mathbf{F} being finite and \mathbf{B} well placed, the homomorphisms $u_i : \mathbf{B} \rightarrow \mathbf{F}$ mapping \bar{a} to $a(i)$ are continuous for all $i < \omega$. It follows that

$$u_i(b) = \lim_{n \rightarrow \infty} u_i(\bar{s}_n) = \lim_{n \rightarrow \infty} s_n(i) = c.$$

On the other hand,

$$\begin{aligned} u_i(b) &= u_i(\overline{t(s_0, \dots, s_k)}) \\ &= t(s_0, \dots, s_k)(i) \\ &= t(s_0(i), \dots, s_k(i)). \end{aligned}$$

Now substituting $i = 1, \dots, k$ into

$$t(s_0(i), \dots, s_k(i)) = c$$

yields the desired near-unanimity equations for t .

It remains to prove that the sequence \bar{s}_n converges pointwise to a continuous function. Fix $x \in X$. We first prove the following claim.

CLAIM. *There is exactly one value that occurs more than once in the sequence $\{\bar{s}_n(x)\}_{n < \omega}$.*

As P is finite there is at least one such value. To see that there can't be more than one, we derive a contradiction from the assumption that

$$\bar{s}_k(x) = \bar{s}_l(x) \neq \bar{s}_m(x) = \bar{s}_n(x),$$

with k, l, m and n pairwise distinct.

Consider the subalgebra \mathbf{C} of \mathbf{A} generated by $\{s_k, s_l, s_m, s_n\}$. Denote by θ the least $[\mathcal{A}-]$ congruence on \mathbf{C} that contains the pairs (s_k, s_l) and (s_m, s_n) , and let ψ denote the least $[\mathcal{A}-]$ congruence on \mathbf{C} that identifies all four generators. We shall prove that $\psi = \theta$ which provides the desired contradiction, for the $[\mathcal{A}-]$ congruence

$$\{(c_1, c_2) \mid \bar{c}_1(x) = \bar{c}_2(x)\}$$

obviously contains θ but not ψ .

Every element of \mathbf{C} can be written in the form $t(s_k, s_l, s_m, s_n)$ for some 4-ary term t in the language of \mathbf{P} . The element $t(s_k, s_l, s_m, s_n)(i)$ of F is one of $t(d, c, c, c)$, $t(c, d, c, c)$, $t(c, c, d, c)$, $t(c, c, c, d)$ or $t(c, c, c, c)$ according to whether i equals k, l, m, n or any other natural number. It follows that for all $i < \omega$,

$$t_1(s_k, s_l, s_m, s_n)(i) = t_2(s_k, s_l, s_m, s_n)(i) \Rightarrow t_1(c, c, c, c) = t_2(c, c, c, c). \tag{*}$$

Let η_i denote the kernel of the projection of \mathbf{C} to \mathbf{F} at the i th coordinate. Then we can interpret (*) as saying $\eta_i \leq \eta_j$ for all $i, j \leq \omega$ with $j \notin \{k, l, m, n\}$. It follows that

$$\eta_k \wedge \eta_l \wedge \eta_m \wedge \eta_n = \bigwedge \{\eta_i \mid i < \omega\}$$

is the identity relation on \mathbf{C} . Therefore,

$$\theta = \theta \vee (\eta_k \wedge \eta_l \wedge \eta_m \wedge \eta_n). \tag{**}$$

Moreover, the implication (*) yields $\theta \vee \eta_i \leq \psi$ for all i . Indeed, $\theta \leq \psi$ is obvious and assuming

$$t_1(s_k, s_l, s_m, s_n) \equiv t_2(s_k, s_l, s_m, s_n) \pmod{\eta_i}$$

we get

$$\begin{aligned} t_1(s_k, s_l, s_m, s_n) &\equiv t_1(s_k, s_k, s_k, s_k) \\ &= t_2(s_k, s_k, s_k, s_k) \\ &\equiv t_2(s_k, s_l, s_m, s_n) \pmod{\psi}. \end{aligned}$$

On the other hand, each η_i identifies at least 3 elements of $\{s_k, s_l, s_m, s_n\}$, with s_i being the possible exception, whence $\psi \leq \theta \vee \eta_i$. Consequently, $\psi = \theta \vee \eta_i$ for all $i < \omega$. By the congruence join-semi-distributivity of the (finite) algebra \mathbf{C} combined

with (**), we finally get

$$\psi = \theta \vee (\eta_k \wedge \eta_l \wedge \eta_m \wedge \eta_n) = \theta$$

as desired. Thus the claim is proved.

We may now define a function $b : X \rightarrow P$ by letting $b(x)$ be the repeated value of the sequence $\bar{s}_n(x)$. Clearly, $b = \lim_{n \rightarrow \infty} \bar{s}_n$. To see that b is continuous, consider any $x \in X$ and choose $m \neq n$ such that

$$b(x) = \bar{s}_m(x) = \bar{s}_n(x) = p \in P.$$

Then, by the continuity of \bar{s}_m and \bar{s}_n , the set

$$U := \bar{s}_m^{-1}(\{p\}) \cap \bar{s}_n^{-1}(\{p\})$$

is an open neighbourhood of x in X . Since $b(U) = \{p\}$, the map b is continuous. □

Recall that $\mathbf{A}^\#$ is a well-placed subalgebra of $C(D(\mathbf{A}), \mathbf{P})$ for all $\mathbf{A} \in \mathcal{A}$. If $\underline{\mathbf{P}}$ admits a duality, then, by Theorem 2.4, the algebra $\mathbf{A}^\#$ is closed in $C(D(\mathbf{A}), \mathbf{P})$, whence Theorem 1.2 and the implication (b) \Rightarrow (a) of Theorem 1.4 follow at once from Theorem 3.1. It remains to prove the implication (a) \Rightarrow (b) of Theorem 1.4.

Assume that $\underline{\mathbf{P}}$ has a $(k + 1)$ -ary near-unanimity term. Let \mathbf{A} be a subalgebra of $C(\mathbf{X}, \mathbf{P})$ for some compact space \mathbf{X} . Taking any $\alpha \in C(\mathbf{X}, \mathbf{P})$ in the pointwise closure of A , we have to show that $\alpha \in A$. For any finite subset Y of X there exists an element of A which agrees with α on Y . In particular, for every point $\mathbf{x} = (x_1, \dots, x_k)$ in X^k , there is some function $a_{\mathbf{x}}$ in A such that $\alpha(x_j) = a_{\mathbf{x}}(x_j)$ for all $j = 1, \dots, k$. As the topology on \mathbf{P} is discrete and α and $a_{\mathbf{x}}$ are continuous, these equalities hold in some neighbourhood of \mathbf{x} in X^k . By compactness, finitely many such neighbourhoods suffice to cover X^k . It follows that there exist $a_1, \dots, a_n \in A$ such that

$$(\dagger) \text{ for all } (x_1, \dots, x_k) \in X^k \text{ there is some } i \in \{1, \dots, n\} \text{ such that } \alpha(x_j) = a_i(x_j) \text{ for all } j = 1, \dots, k.$$

Denote $(a_1(x), \dots, a_n(x)) \in P^n$ by $\mathbf{a}(x)$ and put $B := \{\mathbf{a}(x) \mid x \in X\} \subseteq P^n$. Define $\Phi : B \rightarrow P$ by $\Phi(\mathbf{a}(x)) := \alpha(x)$. To see that Φ is well defined, take x, y such that $\alpha(x) \neq \alpha(y)$. By (\dagger) , there is some i such that

$$a_i(x) = \alpha(x) \neq \alpha(y) = a_i(y),$$

that is, $\mathbf{a}(x) \neq \mathbf{a}(y)$.

On each subset $\{\mathbf{a}(x_1), \dots, \mathbf{a}(x_k)\} \subseteq B$ of size k or less, Φ is equal to a term function, namely a projection. Indeed, by (\dagger) , we have $\alpha(x_j) = a_i(x_j)$ for

$j = 1, \dots, k$, that is, $\Phi(\mathbf{a}(x_j)) = \pi_i(\mathbf{a}(x_j))$, where $\pi_i : B \rightarrow P$ is the i th projection. By Baker and Pixley [1, Theorem 2.1(4)], the near-unanimity term yields a term t such that $\Phi(\mathbf{b}) = t(b_1, \dots, b_n)$ for all $\mathbf{b} = (b_1, \dots, b_n) \in B$. Thus

$$\alpha(x) = \Phi(\mathbf{a}(x)) = t(a_1(x), \dots, a_n(x)) = t(a_1, \dots, a_n)(x)$$

for all $x \in X$, whence $\alpha = t(a_1, \dots, a_n)$ belongs to A . Hence the proof of Theorem 1.4 is complete.

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