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A Coalgebraic View of Heyting Duality

Abstract. We give a coalgebraic view of the restricted Priestley duality between Heyting algebras and Heyting spaces. More precisely, we show that the category of Heyting spaces is isomorphic to a full subcategory of the category of all Γ-coalgebras, based on Boolean spaces, where Γ is the functor which maps a Boolean space to its hyperspace of nonempty closed subsets. As an appendix, we include a proof of the characterization of Heyting spaces and the morphisms between them.

Keywords: Coalgebra, Heyting algebra, Priestley duality.

1. Introduction

Priestley duality relates the category of bounded distributive lattices to the category of Priestley spaces by mapping each bounded distributive lattice L to its ordered space $\mathcal{F}_p(L)$ of prime filters, and mapping each Priestley space X to the bounded distributive lattice $\mathcal{U}^{\mathcal{T}}(X)$ of clopen up-sets of X. When restricted to Heyting algebras and Heyting spaces respectively, these mappings give the restricted Priestley duality for Heyting algebras. We refer the reader to Section 2 for the relevant definitions.

Our main result shows that the category of Heyting spaces is isomorphic to a full subcategory $\mathfrak{H}$ of the category of Γ-coalgebras based on the category $\mathfrak{Z}$ of Boolean spaces. Here the type functor $\Gamma : \mathfrak{Z} \to \mathfrak{Z}$ sends each Boolean space X to the hyperspace of nonempty closed subsets of X. In Section 3 we prove this theorem and give an axiomatisation of the category $\mathfrak{H}$.

The restricted Priestley duality for the category of Heyting algebras dates back to 1974. Although the result is well-known, it appears that no proof exists in the literature. In an appendix we provide a proof of this often quoted result.

The work presented in this paper began as an attempt to give a coalgebraic view of Priestley duality for bounded distributive lattices. Since coalgebras are, in a natural sense, dual to algebras, it seemed reasonable to hope that the category of Priestley spaces might be equivalent to a category

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of coalgebras. Of the difficulties we faced, there is one worth mentioning here. In the general case, the notion of coalgebra homomorphism proved too restrictive. There simply aren't enough coalgebra homomorphisms to account for all the morphisms between Priestley spaces. The restrictiveness of the notion of coalgebra morphism is also evident in other settings. In [6], Gumm describes topological spaces using $\mathcal{F}$-coalgebras, where the functor $\mathcal{F}$ is the filter functor on the category of sets. Here coalgebra morphisms correspond not to continuous maps but to continuous *open* maps. Also, when attempting to describe relational structures as set-based coalgebras in the natural way, one quickly discovers that coalgebra morphisms correspond to relation preserving *and* reflecting maps. These examples suggest that in some settings, a weaker notion of coalgebra morphism might be more appropriate. We suggest that one possibility is to weaken the set equality in the definition to set containment. Despite these difficulties, that our study indicates are inherent in a coalgebraic approach to natural dualities (in the sense of Clark and Davey [2]), Jacobs [7] and Goldblatt [5] have had success in obtaining dualities between certain classes of coalgebras and classes of Boolean algebras with operators.

2. Preliminaries

In this section, we give a brief overview of Heyting algebras, Boolean spaces, Priestley spaces and Heyting spaces. We also give the basic coalgebraic definitions and recall the definition of the hyperspace of a topological space. We refer to Davey and Priestley [3] for notation and facts concerning lattices and ordered sets.

A **Heyting algebra** is an algebra $\langle A; \vee, \wedge, \to, 0, 1\rangle$ of type $\langle 2, 2, 2, 0, 0\rangle$, where $\langle A; \vee, \wedge, 0, 1\rangle$ is a bounded lattice and the binary operation $\to$, which is called **relative pseudocomplementation**, satisfies

$$(\forall a, b, x \in A)\ x \wedge a \leqslant b \Leftrightarrow x \leqslant a \to b.$$

Heyting algebras arise in the algebraic formulation of intuitionistic propositional logic. Note that the operation $\to$ is uniquely determined by the underlying lattice order. The following facts are easily proved: (a) the underlying lattice of a Heyting algebra is necessarily distributive, (b) every finite distributive lattice is (the underlying lattice of) a Heyting algebra, (c) every lattice that satisfies the join-infinite distributive law, $x \wedge \bigvee_{i\in I} y_i = \bigvee_{i\in I} x \wedge y_i$, is a Heyting algebra, (d) every distributive algebraic lattice is a Heyting algebra, (e) the lattice of open subsets of a topological space forms a Heyting algebra.

A **Boolean space** is a compact topological space such that, for all $x, y \in X$ with $x \neq y$, there exists a clopen set V with $x \in V$ and $y \notin V$. We denote the category of Boolean spaces plus continuous maps by $\mathfrak{Z}$. (The letter $\mathfrak{Z}$ is chosen as Boolean spaces are often referred to as compact *zero dimensional* spaces.) A **Priestley space** is a triple $\langle X; \leqslant, \mathcal{T}\rangle$, where $\langle X; \mathcal{T}\rangle$ is a Boolean space and $\leqslant$ is an order relation on X such that, for all $x, y \in X$ with $x \nleqslant y$, there exists a clopen up-set V with $x \in V$ and $y \notin V$. A morphism between Priestley spaces is a continuous order-preserving map. We denote the category of Priestley spaces plus continuous order-preserving maps by $\mathfrak{P}$. For details on Priestley duality see Priestley [9] and Davey and Priestley [3]. Note that for simplicity we will often refer to a Boolean or Priestley space by its underlying set X.

A **Heyting space** X is a Priestley space such that $\downarrow U$ is open for every open subset U of X. (Recall that $\downarrow U := \{\, y \in X \mid (\exists u \in U)\ y \leqslant u \,\}$ and that $\downarrow\{x\}$ is abbreviated to $\downarrow x$. The sets $\uparrow U$ and $\uparrow x$ are defined dually.) A morphism between Heyting spaces, called a **Heyting morphism**, is a continuous map $\varphi : X \to Y$ such that $\varphi(\uparrow x) = \uparrow\varphi(x)$ for all $x \in X$. The **restricted Priestley duality for Heyting algebras** states that a bounded distributive lattice A is the underlying lattice of a Heyting algebra if and only if the Priestley dual of A is a Heyting space, and that a $\{0,1\}$-lattice homomorphism φ between Heyting algebras preserves the operation $\to$ if and only if the Priestley dual of φ is a Heyting morphism. A proof of these facts is presented in Section 4. We denote the category of Heyting spaces plus Heyting morphisms by $\mathfrak{Y}$.

The following proposition records some basic facts about Boolean and Priestley spaces that we shall require. These can be found in [3, Chapter 10].

PROPOSITION 2.1. (1) *Let X be a Boolean space.*

- (a) *X is Hausdorff, and hence, each singleton set $\{x\} \subseteq X$ is closed.*
- (b) *If $V \subseteq X$ is closed and $y \in X$ with $y \notin V$, then there exists a clopen set W with $V \subseteq W$ and $y \notin W$.*

(2) *Let X be a Priestley space.*

- (a) *If $V \subseteq X$ is closed, then $\downarrow V$ and $\uparrow V$ are closed (and in particular, $\downarrow x$ and $\uparrow x$ are closed, for each $x \in X$),*
- (b) *If $U \subseteq X$ is clopen, then there is a finite set $\{\, U_i, V_i \mid i \in I \,\}$ of clopen up-sets of X such that $U = \bigcup_{i \in I}(U_i \setminus V_i)$.*
- (c) *If Y and Z are disjoint closed subsets of X such that Y is an up-set and Z is a down-set, then there exists a clopen up-set W such that $Y \subseteq W$ and $W \cap Z = \varnothing$.*

(3) *Let X and Y be Priestley spaces and let $\varphi : X \to Y$ be a continuous map. Then φ is order-preserving if and only if $\varphi^{-1}(U)$ is a clopen up-set of X, for every clopen up-set U of Y.*

Given a category $\mathfrak{C}$ and a functor $T : \mathfrak{C} \to \mathfrak{C}$, a T-**coalgebra** (in $\mathfrak{C}$) is a pair $\langle A; \alpha_A \rangle$, where A is an object in $\mathfrak{C}$ and $\alpha_A : A \to T(A)$ is a morphism in $\mathfrak{C}$. A T-**homomorphism** of coalgebras $\langle A; \alpha_A \rangle$ and $\langle B; \alpha_B \rangle$ is a morphism $h : A \to B$ in $\mathfrak{C}$ such that $\alpha_B \circ h = T(h) \circ \alpha_A$. The resulting category of T-coalgebras in $\mathfrak{C}$ is denoted $\mathfrak{C}_T$. The functor T called the **type** functor. Coalgebras, particularly in the category of sets, are playing an increasingly important role in theoretical computer science: see Rutten [11] for an excellent introduction to both the general theory and the applications.

If X is a topological space, the **hyperspace** of X is the set $\Gamma(X)$ of all nonempty closed subsets of X. For $V \subseteq X$, we set

$$X_V := \{\, E \in \Gamma(X) \mid E \subseteq V \,\} \qquad \text{and} \qquad X^V := \{\, E \in \Gamma(X) \mid E \cap V \neq \varnothing \,\}.$$

We take the family of sets of the form X_V or X^V, with V open, as a subbasis for a topology on $\Gamma(X)$. This topology has a variety of names in the literature including **Vietoris**, **finite** and **exponential** topology. If V is clopen in X, then the sets X_V and X^V are clopen in $\Gamma(X)$ (see [8]). From this fact and Proposition 2.1(1)(b), it is easily seen that if X is a Boolean space, then $\Gamma(X)$ is also a Boolean space. Furthermore, if $\varphi : X \to Y$ is a continuous map between Boolean spaces, then $\Gamma(\varphi) : \Gamma(X) \to \Gamma(Y)$, given by $\Gamma(\varphi)(A) = \varphi(A)$, is well defined and continuous. Thus, $\Gamma : \mathfrak{Z} \to \mathfrak{Z}$ is a functor. We shall study the category $\mathfrak{Z}_\Gamma$ of Γ-coalgebras in $\mathfrak{Z}$.

Recall that two categories $\mathfrak{A}$ and $\mathfrak{B}$ are said to be **isomorphic** if there exist functors $F : \mathfrak{A} \to \mathfrak{B}$ and $G : \mathfrak{B} \to \mathfrak{A}$ such that $G \circ F = \mathrm{id}_{\mathfrak{A}}$ and $F \circ G = \mathrm{id}_{\mathfrak{B}}$.

3. Heyting spaces and Heyting coalgebras

In this section we prove the main result of this note, namely, that the category $\mathfrak{Y}$ of Heyting spaces is isomorphic to a full subcategory of $\mathfrak{Z}_\Gamma$. We begin by defining a functor from $\mathfrak{Y}$ to $\mathfrak{Z}_\Gamma$. To simplify our notation, for each $V \subseteq X$ we denote the set $X \setminus V$ by V' and abbreviate $\downarrow(V')$ to $\downarrow V'$.

PROPOSITION 3.1. *For a Heyting space X define $F(X) := \langle X; \alpha \rangle$, where $\alpha(x) = \uparrow x$, for all $x \in X$, and for a Heyting morphism $\varphi : X \to Y$, define $F(\varphi) := \varphi$. Then $F : \mathfrak{Y} \to \mathfrak{Z}_\Gamma$ is a functor.*

PROOF. By Proposition 2.1(2)(a) and the reflexivity of the order relation on X, it follows that $\alpha(x) \in \Gamma(X)$, for all $x \in X$. We now show that α is continuous. For each $V \subseteq X$, we have

$$\alpha^{-1}(X_V) = \{\, x \in X \mid \uparrow x \subseteq V \,\} \quad \text{and} \quad \alpha^{-1}(X^V) = \{\, x \in X \mid \uparrow x \cap V \neq \varnothing \,\}.$$

Observe that $\alpha^{-1}(X^V) = \downarrow V$ since

$$z \in \downarrow V \Leftrightarrow z \leqslant v \text{ for some } v \in V \Leftrightarrow v \in \uparrow z \cap V.$$

Furthermore, $\alpha^{-1}(X_V) = (\downarrow V')'$ since

$$\begin{aligned} z \in \alpha^{-1}(X_V) \Leftrightarrow \uparrow z \subseteq V \Leftrightarrow &\uparrow z \cap V' = \varnothing \\ &\Leftrightarrow z \notin \{\, x \mid \uparrow x \cap V' \neq \varnothing \,\} = \downarrow V' \\ &\Leftrightarrow z \in (\downarrow V')'. \end{aligned}$$

Hence, for V open, V' is closed and so $\downarrow V'$ is closed by Proposition 2.1(2)(a), whence $\alpha^{-1}(X_V)$ open. Moreover, since X is a Heyting space, $\downarrow V$ is open, and thus $\alpha^{-1}(X^V)$ open. It follows that α is continuous and so F is well defined. Finally, it is obvious that F preserves composition and identity maps and so is a functor. ∎

PROPOSITION 3.2. *The category $F(\mathfrak{Y})$ is a full subcategory of $\mathfrak{Z}_\Gamma$.*

PROOF. Let X and Y be Heyting spaces. Then,

$$\begin{aligned} \varphi \in \hom_{\mathfrak{Y}}(X,Y) &\Leftrightarrow (\forall x \in X)\ \varphi(\uparrow x) = \uparrow\varphi(x) \ \ \&\ \ \varphi \text{ continuous} \\ &\Leftrightarrow (\forall x \in X)\ \Gamma(\varphi)(\alpha_X(x)) = \alpha_Y(\varphi(x)) \ \ \&\ \ \varphi \text{ continuous} \\ &\Leftrightarrow \Gamma(\varphi) \circ \alpha_X = \alpha_Y \circ \varphi \ \ \&\ \ \varphi \text{ continuous} \\ &\Leftrightarrow \varphi \in \hom_{\mathfrak{Z}_\Gamma}(F(X), F(Y)). \end{aligned}$$

∎

We wish to define a functor G from $F(\mathfrak{Y})$ back onto $\mathfrak{Y}$ and give an axiomatic characterization of the category $F(\mathfrak{Y})$. We begin by considering functors mapping out of the entire category $\mathfrak{Z}_\Gamma$. To provide a codomain for the functors we have in mind, we require a category which is larger than either of the categories $\mathfrak{Y}$ or $\mathfrak{P}$.

We consider the category of all Boolean spaces with one topologically closed binary relation plus continuous relation-preserving maps. We denote this category by $\mathfrak{Z}_{\mathrm{rel}}$. It is clear that both $\mathfrak{Y}$ and $\mathfrak{P}$ are subcategories of $\mathfrak{Z}_{\mathrm{rel}}$.

Let $\langle X; \alpha\rangle \in \mathfrak{Z}_\Gamma$ and define binary relations R and S on X by

$$\begin{aligned} x\,R\,y &\quad \text{if and only if} \quad y \in \alpha(x), \\ x\,S\,y &\quad \text{if and only if} \quad \alpha(y) \subseteq \alpha(x). \end{aligned}$$

PROPOSITION 3.3. *The mappings $G_R : \mathfrak{Z}_\Gamma \to \mathfrak{Z}_{\mathrm{rel}}$ and $G_S : \mathfrak{Z}_\Gamma \to \mathfrak{Z}_{\mathrm{rel}}$ defined on objects by $G_R(\langle X;\alpha\rangle) = \langle X;R\rangle$ and $G_S(\langle X;\alpha\rangle) = \langle X;S\rangle$, and defined on morphisms by $G_R(\varphi) = G_S(\varphi) = \varphi$ are functors.*

PROOF. It is obvious that these mappings are well defined on objects and preserve composition and identity maps. We need to show that they are well defined on morphisms. Let $\varphi : X \to Y$ be a coalgebra homomorphism between Γ-coalgebras $\langle X;\alpha_X\rangle$ and $\langle Y;\alpha_Y\rangle$ and let $x\,R\,y$. Then $y \in \alpha_X(x)$ and so $\varphi(y) \in \Gamma(\varphi)(\alpha_X(x))$, giving $\varphi(y) \in \alpha_Y(\varphi(x))$ and hence, $\varphi(x)\,R\,\varphi(y)$. If $x\,S\,y$, then $\alpha_X(y) \subseteq \alpha_X(x)$ and hence $\varphi(\alpha_X(y)) \subseteq \varphi(\alpha_X(x))$. Consequently, $\alpha_Y(\varphi(y)) \subseteq \alpha_Y(\varphi(x))$ and thus $\varphi(x)\,S\,\varphi(y)$. It follows that G_R and G_S are well defined on morphisms. ∎

Given $\langle X;\alpha\rangle \in \mathfrak{Z}_\Gamma$, the relation S on X is a quasi-order (that is, S is reflexive and transitive) and so is a natural choice when trying to relate α to the order on a Priestley space. We now show that the relation R is a natural choice when relating α to the topology. For each $V \subseteq X$, we set

$$\downarrow_R V = \{\, x \in X \mid (\exists v \in V)\; x\,R\,v \,\} \quad\text{and}\quad \uparrow_R V = \{\, x \in X \mid (\exists v \in V)\; v\,R\,x \,\}.$$

The next result illustrates the connection between these sets and the coalgebra structure map α.

LEMMA 3.4. *Let X be a Boolean space, let $\alpha : X \to \Gamma(X)$ be a map and let $V \subseteq X$. Then,*

$$\alpha^{-1}(X^V) = \downarrow_R V, \quad \alpha^{-1}(X_V) = (\downarrow_R V')', \quad \uparrow_R V = \bigcup_{v\in V} \alpha(v).$$

PROOF. We have

$$\alpha^{-1}(X^V) = \{\, x \in X \mid \alpha(x) \cap V \neq \varnothing \,\} = \{\, x \in X \mid (\exists v \in V)\; x\,R\,v \,\} = \downarrow_R V,$$

and

$$\alpha^{-1}(X_V) = \{\, x \in X \mid \alpha(x) \subseteq V \,\} = \{\, x \in X \mid \alpha(x) \cap V' \neq \varnothing \,\}' = (\downarrow_R V')',$$

where the last equality follows from the previous calculation. Finally,

$$x \in \uparrow_R V \Leftrightarrow (\exists v \in V)\; v\,R\,x \Leftrightarrow (\exists v \in V)\; x \in \alpha(v) \Leftrightarrow x \in \bigcup_{v\in V} \alpha(v),$$

as required. ∎

The significance of Lemma 3.4 is that it characterizes the continuity of the coalgebra structure map α in terms of the down-sets with respect to the relation R. We record this in the next corollary.

COROLLARY 3.5. *Let X be a Boolean space and let $\alpha : X \to \Gamma(X)$ be a map. Then α is continuous if and only if the following conditions hold:*

(1) $(\forall V \subseteq X)$ *V open in $X \Rightarrow {\downarrow_R} V$ open in X,*
(2) $(\forall V \subseteq X)$ *V closed in $X \Rightarrow {\downarrow_R} V$ closed in X.*

PROOF. Assume that α is continuous. If V is open in X, then $\alpha^{-1}(X^V) = {\downarrow_R} V$ is open in X by Lemma 3.4. If V is closed in X, then V' is open in X and so $\alpha^{-1}(X_{V'}) = ({\downarrow_R} V'')' = ({\downarrow_R} V)'$ is open in X, by Lemma 3.4. Hence, ${\downarrow_R} V$ is closed in X. Conversely, assume (1) and (2) hold and let V be open in X. Then, $\alpha^{-1}(X^V) = {\downarrow_R} V$ and $\alpha^{-1}(X_V) = ({\downarrow_R} V')'$ are open in X, and hence α is continuous. ■

The next result illustrates the usefulness of Corollary 3.5.

PROPOSITION 3.6. *Let $\langle X; \alpha \rangle$ be a Γ-coalgebra in $\mathfrak{Z}$. The following are equivalent:*

(1) *$G_R(\langle X; \alpha \rangle)$ is a Heyting space;*
(2) *$G_R(\langle X; \alpha \rangle)$ is a Priestley space;*
(3) *R is an order relation.*

PROOF. Certainly, (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Assume that R is an order. By the continuity of α and Corollary 3.5, we have ${\downarrow_R} U$ open for each open set U. Therefore, to prove (1), it suffices to show that $G_R(\langle X; \alpha \rangle)$ is a Priestley space. Let $x, y \in X$ with $(y, x) \notin R$. Then $x \neq y$, by reflexivity, and $x \notin \alpha(y)$. Since X is a Boolean space and $\alpha(y)$ is closed, there exists a clopen set V with $x \in V$ and $V \cap \alpha(y) = \varnothing$. By Corollary 3.5, ${\downarrow_R} V$ is clopen. Furthermore, $x \in {\downarrow_R} V$ and clearly $y \notin {\downarrow_R} V$, since otherwise we would have $v \in \alpha(y)$ for some $v \in V$ contradicting $V \cap \alpha(y) = \varnothing$. ■

This result shows that G_R maps into $\mathfrak{Y}$ precisely when R is an order relation. The functor G_S is useful for identifying ordered Boolean spaces amongst the objects of $\mathfrak{Z}_{\text{rel}}$. Indeed, for all $\langle X; \alpha \rangle \in \mathfrak{Z}_\Gamma$, the relation S on X is reflexive and transitive, and is anti-symmetric if and only if α is one-to-one. Thus, Propositions 3.6 shows that G_R maps into $\mathfrak{Y}$ provided $R = S$ and the map α is one-to-one. It is easy to see that, in general, we need not have $R = S$. For example, consider the set $X = \{1, 2\}$ with the discrete topology and the structure map $\alpha : X \to \Gamma(X)$ given by $1 \mapsto \{2\}$, $2 \mapsto \{1\}$. Equality can be forced, however, using Condition (b) in the following definition.

A Γ-coalgebra $\langle X; \alpha \rangle$ is said to be a **Heyting coalgebra** if

(a) α is one-to-one, and
(b) $y \in \alpha(x) \Leftrightarrow \alpha(y) \subseteq \alpha(x)$, for all $x, y \in X$.

We remark that Condition (b) is equivalent to the following two conditions:

(b)$_1$ $x \in \alpha(x)$, for all $x \in X$, and

(b)$_2$ $y \in \alpha(x) \Rightarrow \alpha(y) \subseteq \alpha(x)$, for all $x, y \in X$.

We denote by $\mathfrak{H}$ the full subcategory of $\mathfrak{Z}_\Gamma$ whose objects are the Heyting coalgebras.

Since $G_R{\restriction}_{\mathfrak{H}} = G_S{\restriction}_{\mathfrak{H}}$ and both restrictions map into $\mathfrak{Y}$, we drop the subscripts and write simply $G : \mathfrak{H} \to \mathfrak{Y}$. The functors F and G now give the isomorphism we seek.

THEOREM 3.7. *The categories $\mathfrak{Y}$ of Heyting spaces and $\mathfrak{H}$ of Heyting coalgebras are isomorphic via the mutually inverse functors $F : \mathfrak{Y} \to \mathfrak{H}$ and $G : \mathfrak{H} \to \mathfrak{Y}$.*

PROOF. Let $\langle X; \leqslant, \mathcal{T}\rangle \in \mathfrak{Y}$, let α be the map obtained via $F(\langle X; \leqslant, \mathcal{T}\rangle)$ and let R be the ordering obtained via $GF(\langle X; \leqslant, \mathcal{T}\rangle)$. We have $x\,R\,y \Leftrightarrow y \in \alpha(x) \Leftrightarrow x \leqslant y$. Hence $R = \leqslant$, giving $GF(\langle X; \leqslant, \mathcal{T}\rangle) = \langle X; \leqslant, \mathcal{T}\rangle$. Let $\langle X; \alpha\rangle \in \mathfrak{H}$, let R denote the order obtained via $G(\langle X; \alpha\rangle)$ and let α' be the map obtained via $FG(\langle X; \alpha\rangle)$. For each $x \in X$, we have $y \in \alpha'(x) \Leftrightarrow x\,R\,y \Leftrightarrow y \in \alpha(x)$. Hence $\alpha' = \alpha$, giving $FG(\langle X; \alpha\rangle) = \langle X; \alpha\rangle$. Finally, we have $GF(\varphi) = FG(\varphi) = \varphi$, for each morphism φ, and so $G \circ F = \mathrm{id}_{\mathfrak{Y}}$ and $F \circ G = \mathrm{id}_{\mathfrak{H}}$. ∎

4. Appendix: a long-awaited proof

The description of the restricted Priestley duality for Heyting algebras was first worked out in 1974 by M. Adams. The paper in which the description appeared was distributed to a number of those working on applications of Priestley duality but was never published. The first time it appears in print seems to be ten years later in Priestley's survey article [10], followed soon after by Adams, Koubek and Sichler [1]. Then and since, all references to the restricted Priestley duality for Heyting algebras refer to it as *folklore*. In fact, also in 1994, Èsakia [4] published a description of Heyting spaces and their morphisms. Èsakia gives a duality for Boolean algebras with an additional closure operation and then indicates how to use this duality to obtain a duality for Heyting algebras. It should be noted that Èsakia's result was obtained without reference to either distributive lattices or Priestley duality. Indeed, because the proof of his duality for Heyting algebras is indirect and missing many details, it is not immediately clear from the paper that Èsakia's duality actually is the restricted Priestley duality.

To complete our coalgebraic excursion into Heyting algebras we shall give a self-contained proof that Heyting spaces and their morphisms are precisely the Priestley duals of Heyting algebras and their homomorphisms. The first step is a very simple set-theoretic fact about up-sets that tells us that the relative pseudocomplement in the lattice of all up-sets of an ordered set X is given by $U \to V = X \setminus {\downarrow}(U \setminus V)$.

LEMMA 4.1. *Let X be an ordered set. For all up-sets U, V and W of X, we have*

$$W \cap U \subseteq V \Leftrightarrow W \subseteq X \setminus {\downarrow}(U \setminus V).$$

PROOF. Let $W \in \mathcal{U}^{\mathcal{T}}(X)$. Assume that $W \cap U \subseteq V$ and let $w \in W$. Suppose $w \in {\downarrow}(U \setminus V)$. Then $w \leqslant u$ for some $u \in U \setminus V$, giving $u \in W$, since W is an up-set. This gives $u \in V$ contradicting $u \in U \setminus V$. Hence, $w \in X \setminus {\downarrow}(U \setminus V)$. Conversely, assume $W \subseteq X \setminus {\downarrow}(U \setminus V)$ and let $y \in W \cap U$. Suppose $y \notin V$. Then, $y \in U \setminus V$, giving $y \in W \cap {\downarrow}(U \setminus V)$ which is a contradiction. Hence $y \in V$. ■

Recall that a subset Q of an ordered set X is **convex** if, for all $a, b \in Q$ and $x \in X$, $a \leqslant x \leqslant b$ implies $x \in Q$, and that the lattice of clopen up-sets of an ordered topological space X is denoted here by $\mathcal{U}^{\mathcal{T}}(X)$. Condition (1) in the following theorem is Èsakia's description of Heyting spaces while Condition (3) is Adams' description.

FOLKLORE THEOREM 4.2. *Let X be an ordered Boolean space. Then the following are equivalent:*

(1) *${\uparrow}x$ is closed, for all $x \in X$, and ${\downarrow}U$ is clopen, for all clopen subsets U of X;*
(2) *X is a Priestley space such that ${\downarrow}U$ is open, for all open subsets U of X;*
(3) *X is a Priestley space such that ${\downarrow}U$ is open, for all convex open subsets U of X;*
(4) *X is a Priestley space such that ${\downarrow}(U \setminus V)$ is clopen, for all $U, V \in \mathcal{U}^{\mathcal{T}}(X)$;*
(5) *X is a Priestley space and $\mathcal{U}^{\mathcal{T}}(X)$ is a Heyting algebra in which*

$$U \to V = X \setminus {\downarrow}(U \setminus V),$$

for all $U, V \in \mathcal{U}^{\mathcal{T}}(X)$;
(6) *X is a Priestley space and $\mathcal{U}^{\mathcal{T}}(X)$ is a Heyting algebra.*

PROOF. We prove (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1) and (4) $\Rightarrow$ (5) $\Rightarrow$ (6) $\Rightarrow$ (4).

(1) $\Rightarrow$ (2): Assume (1). Since every open set in X is a union of clopen sets, to prove (2) it suffices to show that X is a Priestley space. Let $x, y \in X$ with $x \nleqslant y$. Then $X \setminus {\uparrow}x$ is an open neighbourhood of y and, since X is a

Boolean space, there exists a clopen set U with $y \in U \subseteq X \setminus {\uparrow}x$. By (1), the set $V := X \setminus {\downarrow}U$ is a clopen up-set that contains x but not y. Hence, X is a Priestley space.

(2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (4): Assume (3) holds and let $U, V \in \mathcal{U}^{\mathcal{T}}(X)$. Since $U \setminus V = U \cap V'$ is clopen, ${\downarrow}(U \setminus V)$ is closed by Proposition 2.1(2)(a). Since U is an up-set and V' is a down-set, it follows at once that $U \setminus V = U \cap V'$ is convex. Hence, from (3) we have ${\downarrow}(U \setminus V)$ is open, and therefore clopen.

(4) $\Rightarrow$ (1): Assume (4) holds. As X is a Priestley space, by Proposition 2.1(2)(a), the set ${\uparrow}x$ is closed, for all $x \in X$. Now let U be clopen in X. By Proposition 2.1(2)(b), there is a finite set I and U_i, V_i in $\mathcal{U}^{\mathcal{T}}(X)$, for each $i \in I$, such that $U = \bigcup_{i \in I}(U_i \setminus V_i)$. Since, by (4), ${\downarrow}(U_i \setminus V_i)$ is clopen, for all $i \in I$, and since I is finite, it follows that ${\downarrow}U$ is clopen.

(4) $\Rightarrow$ (5): Assume (4) holds and let $U, V \in \mathcal{U}^{\mathcal{T}}(X)$. Set $C = X \setminus {\downarrow}(U \setminus V)$ and observe that by Lemma 4.1, it suffices to show that $C \in \mathcal{U}^{\mathcal{T}}(X)$. Observe that C is an up-set since it is the complement of a down-set. Furthermore, $U \setminus V$ is clopen since U and V are clopen. Since (4) $\Rightarrow$ (1), from above, ${\downarrow}(U \setminus V)$ is clopen. Hence, C is a clopen up-set of X.

(5) $\Rightarrow$ (6) is trivial.

(6) $\Rightarrow$ (4): Assume that (6) holds and let $U, V \in \mathcal{U}^{\mathcal{T}}(X)$. Then there exists $C \in \mathcal{U}^{\mathcal{T}}(X)$ such that

$$W \cap U \subseteq V \Leftrightarrow W \subseteq C$$

for all $W \in \mathcal{U}^{\mathcal{T}}(X)$. Observe that if $X \setminus {\downarrow}(U \setminus V) \in \mathcal{U}^{\mathcal{T}}(X)$, then ${\downarrow}(U \setminus V)$ is clopen. Therefore, it suffices to show that $X \setminus {\downarrow}(U \setminus V) = C$. Now, since $C \in \mathcal{U}^{\mathcal{T}}(X)$ and $C \subseteq C$, we have $C \cap U \subseteq V$. Applying Lemma 4.1 gives $C \subseteq X \setminus {\downarrow}(U \setminus V)$. Suppose that the containment is strict. That is, there exists $y \in X$ with $y \in X \setminus {\downarrow}(U \setminus V)$ and $y \notin C$. Since U and V are clopen, $U \setminus V$ is clopen. Therefore, ${\downarrow}(U \setminus V)$ is a closed down-set, by Proposition 2.1(2)(a). Similarly, ${\uparrow}y$ is a closed up-set. Moreover, since $y \notin {\downarrow}(U \setminus V)$, we have ${\uparrow}y \cap {\downarrow}(U \setminus V) = \varnothing$. Thus, by Proposition 2.1(2)(c), there exists $W \in \mathcal{U}^{\mathcal{T}}(X)$ with $y \in W$ and $W \subseteq X \setminus {\downarrow}(U \setminus V)$. Applying Lemma 4.1 gives $W \cap U \subseteq V$, and consequently, $W \subseteq C$ since $W \in \mathcal{U}^{\mathcal{T}}(X)$ and C satisfies the relative pseudocomplementation property above. Finally, since $y \in W$, this gives $y \in C$ which is a contradiction. ∎

We turn now to the description of the Priestley duals of Heyting algebra homomorphisms. Condition (6) is the folkloric description.

FOLKLORE THEOREM 4.3. *Let $\varphi : X \to Y$ be a continuous map between Heyting spaces X and Y. Then the following are equivalent:*

(1) $\varphi^{-1} : \mathcal{U}^{\mathcal{T}}(Y) \to \mathcal{U}^{\mathcal{T}}(X)$ *is a well-defined Heyting homomorphism;*

(2) $\varphi^{-1}(\downarrow(U \setminus V)) = \downarrow\varphi^{-1}(U \setminus V)$, *for all* $U, V \in \mathcal{U}^{\mathcal{T}}(Y)$*;*

(3) $\varphi^{-1}(\downarrow U) = \downarrow\varphi^{-1}(U)$, *for all clopen subsets* U *of* Y*;*

(4) $\varphi^{-1}(\downarrow S) = \downarrow\varphi^{-1}(S)$, *for every subset* S *of* Y*;*

(5) $\varphi^{-1}(\downarrow y) = \downarrow\varphi^{-1}(y)$, *for all* $y \in Y$*;*

(6) $\varphi(\uparrow x) = \uparrow\varphi(x)$, *for all* $x \in X$.

PROOF. We shall prove (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3).

(1) $\Leftrightarrow$ (2): Assume that $\varphi^{-1} : \mathcal{U}^{\mathcal{T}}(Y) \to \mathcal{U}^{\mathcal{T}}(X)$ is a well defined. Then φ^{-1} certainly preserves $\cup$, $\cap$, $\bot$ and $\top$. Since, by Folklore Theorem 4.2, $U \to V$ is given in $\mathcal{U}^{\mathcal{T}}(Y)$ by $Y \setminus \downarrow(U \setminus V)$, it is easily seen that φ^{-1} preserves the operation $\to$ if and only if $\varphi^{-1}(\downarrow(U \setminus V)) = \downarrow\varphi^{-1}(U \setminus V)$, for all U, V in $\mathcal{U}^{\mathcal{T}}(Y)$. Thus, to complete the proof of (1) $\Leftrightarrow$ (2), it remain to show that (2) implies that $\varphi^{-1} : \mathcal{U}^{\mathcal{T}}(Y) \to \mathcal{U}^{\mathcal{T}}(X)$ is well defined, that is, that $\varphi^{-1}(U)$ is a clopen up-set for every clopen up-set U of Y. Let U be a clopen up-set of Y. Then

$$
\begin{aligned}
\varphi^{-1}(U) &= \varphi^{-1}(Y \setminus (Y \setminus U)) \\
&= X \setminus \varphi^{-1}(\downarrow(Y \setminus U)) \quad \text{as } Y \setminus U \text{ is a down-set} \\
&= X \setminus \downarrow\varphi^{-1}(Y \setminus U) \qquad \text{by (2).}
\end{aligned}
$$

Hence $\varphi^{-1}(U)$ is an up-set since $\downarrow\varphi^{-1}(Y \setminus U)$ is a down-set. (Of course, $\varphi^{-1}(U)$ is clopen as φ is continuous.)

(2) $\Rightarrow$ (3) follows easily from the fact (see Proposition 2.1(2)(b)) that every clopen subset of Y is a union of clopen sets of the form $U \setminus V$ with $U, V \in \mathcal{U}^{\mathcal{T}}(Y)$, and (3) $\Rightarrow$ (2) is trivial.

(3) $\Rightarrow$ (6): Assume (3) holds. Since (3) $\Rightarrow$ (1), the map $\varphi^{-1} : \mathcal{U}^{\mathcal{T}}(Y) \to \mathcal{U}^{\mathcal{T}}(X)$ is well defined, and hence by Proposition 2.1(3), φ is order-preserving. Let $x \in X$. Since φ is order-preserving, we have $\varphi(\uparrow x) \subseteq \uparrow\varphi(x)$. Let $y \in \uparrow\varphi(x)$. Since $\uparrow x$ is closed in X (by Proposition 2.1(2)(a)) and φ is a closed map, $\varphi(\uparrow x)$ is closed in Y. Thus, to prove that $y \in \varphi(\uparrow x)$, it suffices to show that every clopen set U that contains y intersects $\varphi(\uparrow x)$. Let U be a clopen subset of Y with $y \in U$. Since $\varphi(x) \leqslant y$, we have $x \in \varphi^{-1}(\downarrow U) = \downarrow\varphi^{-1}(U)$, by (3), whence there exists $z \in X$ with $x \leqslant z$ and $\varphi(z) \in U$. It follows that $\varphi(z) \in \varphi(\uparrow x) \cap U$, giving $\varphi(\uparrow x) \cap U \neq \varnothing$, as required.

(6) $\Rightarrow$ (5): Assume (6) and let $y \in Y$. Then, by (6),

$$
\begin{aligned}
x \in \varphi^{-1}(\downarrow y) &\Leftrightarrow \varphi(x) \leqslant y \\
&\Leftrightarrow y \in \uparrow\varphi(x) = \varphi(\uparrow x)
\end{aligned}
$$

$$\Leftrightarrow (\exists t \in {\uparrow}x)\ \varphi(t) = y$$
$$\Leftrightarrow x \in {\downarrow}\varphi^{-1}(y),$$

giving $\varphi^{-1}({\downarrow}y) = {\downarrow}\varphi^{-1}(y)$.

(5) $\Rightarrow$ (4) $\Rightarrow$ (3): Since both φ^{-1} and $\downarrow$ preserve arbitrary unions, (4) is an easy consequence of (5), and, of course, (4) $\Rightarrow$ (3) is trivial. ∎

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