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When is a natural duality 'good'?

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Dedicated to the memory of Alan Day

Abstract. Building on the most current work in the theory of natural dualities, we continue the study of strong dualities for the quasi-variety generated by a finite algebra. We investigate ten different versions of what we would like to mean by a 'good duality'. Each version concerns, among other things, a specific restriction on the type of the structures in the dual category which insures that the dual structures will in a useful sense be simple. Through each investigation we seek a theorem characterizing, in terms of finitely verifiable conditions, those finite algebras generating a quasi-variety which admits a strong duality meeting the given restrictions. Our study includes a careful treatment of coproducts, logarithmic dualities and strong dualities by various unary structures.

0. Introduction

This paper concerns a class of topological dualities for finitely generated quasi-varieties $\mathscr{A} = \mathbb{ISP}\underline{\mathbf{M}}$ which has proven to be broad enough to emcompass many interesting examples yet narrow enough to enjoy a deep and powerful theory. Such dualities, known as *natural dualities*, were originally developed in Davey and Werner [13] and Clark and Krauss [7]. Recently the general theory of natural dualities has been updated in Davey [9] and further expanded in Clark and Davey [5]. We outline briefly the context of this theory, postponing careful definitions and references to Section 1. We begin by imposing a new structure $\underset{\sim}{\mathbf{M}}$, endowed with the discrete topology, on the carrier M of the finite generating algebra $\underline{\mathbf{M}}$ and using it to generate a new category $\mathscr{X} = \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$ of compact topological structures. In general the structure $\underset{\sim}{\mathbf{M}}$ may have total operations, partial operations and relations. If $\underset{\sim}{\mathbf{M}}$ is chosen properly, then, for each $\mathbf{X} \in \mathscr{X}$, the set of all continuous homomorphisms

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from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$ will form a subalgebra $E(\mathbf{X})$ of $\underline{\mathbf{M}}^X$ and E itself will be a contravariant functor from $\mathscr{X}$ to $\mathscr{A}$. With even more care we find that $\underset{\sim}{\mathbf{M}}$ can be chosen in such a way that every member of $\mathscr{A}$ will be isomorphic to $E(\mathbf{X})$ for some $\mathbf{X}$ in $\mathscr{X}$. When this kind of representation of algebras in $\mathscr{A}$ arises in a uniform manner we say that $\mathbf{M}$ *yields a (natural) duality on* $\mathscr{A}$.

Once we begin to look, we find that dualities of this sort do exist in amazing profusion. For this reason it is important to determine which ones constitute 'good dualities'. Examples of dualities that have historically generated the most interest are those in which the contravariant functor $E: \mathscr{X} \to \mathscr{A}$ is a full equivalence between the categories $\mathscr{X}$ and $\mathscr{A}$, leading to a tight transfer of information between the two categories. The central thesis of Clark and Davey [5] is that there is only one avenue by which any such natural duality has ever been shown to be an equivalence between the categories in question. When this avenue is available we say that $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$.

The general theory developed in Clark and Davey [5] provides the means to generate many complex strong dualities which, in particular, include all previously known instances of natural dualities. Here is one example. Let $\underline{\mathbf{M}}$ be the lattice n-chain for any $n \geq 2$. Then Con $\underline{\mathbf{M}}$ is a Boolean lattice of order 2^{n-1}. Using the NU-Strong Duality Theorem [5, 4.7], we obtain a strong duality for the variety $\mathscr{A}$ of distributive lattices with $\underline{\mathbf{M}}$ as generating algebra by taking $\underset{\sim}{\mathbf{M}} = \langle M; H, R, \mathscr{T} \rangle$ where H, R and $\mathscr{T}$ are chosen as follows: H is the set of all homomorphisms from sublattices of powers $\underline{\mathbf{M}}^k$ into $\underline{\mathbf{M}}$, for some $k < n$, each homomorphism being viewed as a partial operation on M, while R is (a generating set for) the sublattices of $\underline{\mathbf{M}}^2$ viewed as binary relations on M, and $\mathscr{T}$ is the discrete topology on M. The dual category $\mathscr{X}$ is a category of compact structures having the same type as $\underset{\sim}{\mathbf{M}}$.

This example illustrates the central point of this study. What we have is an equivalence between a category of algebras with a relatively simple type, distributive lattices in this instance, and a category $\mathscr{X}$ of topological spaces carrying a very complicated structure. Such a bizarre strong duality would never have arisen out of a study of distributive lattices per se, but only, as it did, out of a much more general theory which yields it as a special case. Whatever we might mean by a 'good duality', this is certainly not an example of one! On the other hand, there are many cases in the literature where substantial applications of strong dualities have been achieved by studying a simple and tractable dual category $\mathscr{X}$ and then transferring information, via the duality, back to $\mathscr{A}$. (See, for example, Stone [14], Priestley [15], Clark [4], Clark and Schmid [8], Davey, Quackenbush and Schweigert [12], Adams and Clark [1].)

To begin with we shall follow the theme of our previous paper [5] and require that a duality be *strong* in order to be good. Surveying the above cited applications we can see, at least in a general conceptual sense, what more needs to be required of a 'good duality'. In each of these cases the dual category $\mathscr{X}$ is in some tangable sense simpler than the algebraic quasi-variety $\mathscr{A}$. Since the two categories are generated in a similar way by structures $\underline{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}$ having the same carrier M, the difference in their complexity depends on the type of the structure $\underset{\sim}{\mathbf{M}}$. Useful dualities have consistently proven to be those in which the dual category $\mathscr{X}$ consists of structures whose type is in some meaningful sense simple.

The goal of this paper is that of determining which choices of $\underline{\mathbf{M}}$ admit a strong duality that is 'good' in the sense that the dual category $\mathscr{X}$ has a type which is restricted in one of several ways that insure its members will be relatively simple. As we will see, these dual categories are often the ones that have other nice properties as well. For example coproducts, which always exist in $\mathscr{X}$ under a full duality, are often much simpler and more easily recognized than the corresponding direct products in $\mathscr{A}$ if the type of $\mathscr{X}$ is suitably restricted. This can lead to illuminating information about congruences on $\mathbf{A} = E(\mathbf{X})$ since they, as subalgebras of $\mathbf{A} \times \mathbf{A}$, are dual to special images of the coproduct $\mathbf{X} * \mathbf{X}$. We shall consider a number of different versions of 'good' by imposing different restrictions on the type of $\mathscr{X}$ which, based on examples in the literature, appear to lead to a useful duality. In each case we would like to know exactly which finitely generated quasi-varieties $\mathscr{A}$ admit a strong duality which is 'good' in the sense in question.

We will prove ten different theorems corresponding to ten different senses in which a strong duality can be considered 'good'. Eight of these theorems are named after their specific restriction on the type of structure $\underset{\sim}{\mathbf{M}}$. Each of our ten theorems gives several equivalent conditions, beginning with the desired restriction on the type of $\underset{\sim}{\mathbf{M}}$ and ending with the corresponding restriction on $\underline{\mathbf{M}}$. Our results are summarized in Figure 1, where we have ordered them by increasing strength of their hypotheses. Here we denote by $\# G$, $\# H$ and $\# R$ the maximum arity of their respective members. For a full statement of each result, the reader is referred to the section indicated.

We are by no means prepared to claim that these results give a complete answer to the question, 'What makes a good duality?' Other reasonable restrictions are available for investigation, and useful additional equivalences to the ones that we have found might still surface. We therefore hope that our efforts will serve to stimulate further investigation into the next question, 'What else makes a good duality?'

1. Preliminaries

We would like to address the algebraist who might bring to us a favorite finitely generated quasi-variety $\mathscr{A} = \mathbb{ISP}\underline{\mathbf{M}}$ and ask if it admits a 'good duality' in the sense of one of our ten theorems. With rather little background it is entirely possible to understand and apply these theorems in order to determine which, if any, of these 'good dualities' exits for the quasi-variety $\mathscr{A}$. In this section we will present that

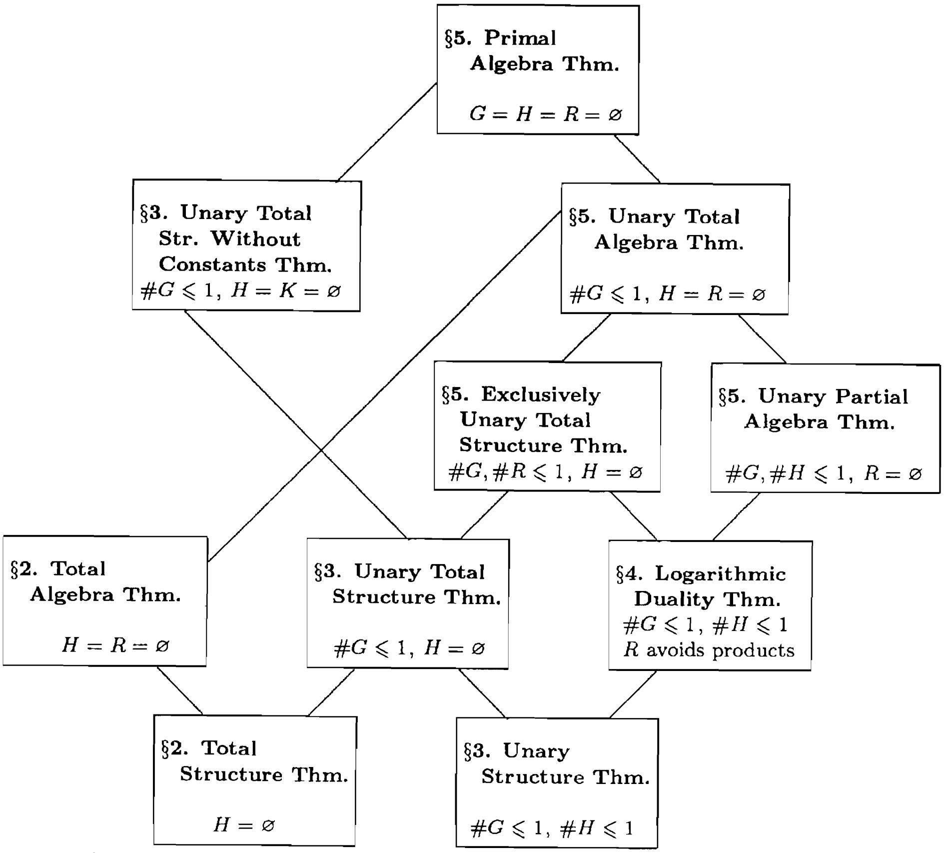

Figure 1. Ten ways a strong duality can be 'good'.

background. On the other hand the proofs of these theorems depend heavily upon recently developed results of the general theory of natural dualities, all of which may be found in Clark and Davey [5]. (See also Davey [9], Davey and Werner [13], or Clark and Krauss [7].) This means that a full understanding of the details will depend upon a familiarity with some of this earlier work. In the near future a full exposition of the theory of natural dualities will be compiled in the authors' comprehensive text [6].

We begin by imagining that we have fixed a choice of a finite algebra $\underline{\mathbf{M}}$ generating a quasi-variety $\mathscr{A} = \mathbb{ISP}\underline{\mathbf{M}}$. In this description of $\mathscr{A}$ we include the power $\underline{\mathbf{M}}^{\varnothing}$, the one element algebra having the type of $\underline{\mathbf{M}}$, which we will denote by $\underline{\mathbf{1}}$. Living on the same carrier M we consider a dual structure

$$\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathscr{T} \rangle$$

where G is a set of total operations, H is a set of partial operations, R is a set of finitary relations and $\mathscr{T}$ is the discrete topology on M. We use $\underset{\sim}{\mathbf{M}}$ to generate the category $\mathscr{X} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$ of isomorphic copies of (possibly empty) topologically closed substructures of direct powers indexed over *nonempty* index sets of the structure $\underset{\sim}{\mathbf{M}}$. In particular, the members of $\mathscr{X}$ are Boolean spaces carrying a structure of the same type as $\underset{\sim}{\mathbf{M}}$. Morphisms are continuous functions which preserve operations, partial operations and relations. Distinguished elements of the structure $\underset{\sim}{\mathbf{M}}$ will play an important role in our constructions. We will view them formally as the values of nullary operations which we include in G. Notice that $\mathscr{X}$ contains the empty structure $\underset{\sim}{\varnothing}$ exactly when there are no nullary operation symbols in its type. The topological categories $\mathscr{X}$ are more fully described in Clark and Davey [5] and extensively studied in Clark and Krauss [7].

Our conventions regarding the empty set have been carefully chosen to make our development run more smoothly. For example, by allowing the empty substructure $\underset{\sim}{\varnothing}$ we insure that the substructures of a member of $\mathscr{X}$ are always closed under intersection and will therefore form a lattice. Those readers preferring to study the universal Horn class $\mathscr{A}^+ = \mathbb{I}\mathbb{S}\mathbb{P}^+\underline{\mathbf{M}}$ may do so, as we ourselves have often done in the past, by omitting $\underset{\sim}{\varnothing}$ from $\mathscr{X}$ and making appropriate minor adjustments to the theory.

The general concept of natural dualities was initiated by an observation of Davey and Werner [13] which gives a simple condition on $\underset{\sim}{\mathbf{M}}$ that will guarantee a tight connection between the categories $\mathscr{A}$ and $\mathscr{X}$. We say that $\underset{\sim}{\mathbf{M}}$ is *algebraic over* $\underline{\mathbf{M}}$ if each operation, partial operation and relation of $\underset{\sim}{\mathbf{M}}$ is a subalgebra of a power of $\underline{\mathbf{M}}$ and each distinguished element of $\underset{\sim}{\mathbf{M}}$ determines a one element subalgebra of $\underline{\mathbf{M}}$. Notice that a (partial) n-ary operation on M is a subalgebra of $\underline{\mathbf{M}}^{n+1}$ if and only if it is a homomorphism from (a subalgebra of) $\underline{\mathbf{M}}^n$ into $\underline{\mathbf{M}}$. This assertion is equivalent to the assertion that the operations on $\underline{\mathbf{M}}$ convert $\underset{\sim}{\mathbf{M}}$ into an $\mathscr{A}$-algebra in X, that is, each operation (and consequently each term function) of $\underline{\mathbf{M}}$ is a morphism from a power of $\underset{\sim}{\mathbf{M}}$ into $\underset{\sim}{\mathbf{M}}$, and each distinguished element of $\underline{\mathbf{M}}$ is the image of $\underset{\sim}{\mathbf{M}}$ under some morphism.

Throughout this study we will only consider choices of $\underset{\sim}{\mathbf{M}}$ that are algebraic over $\underline{\mathbf{M}}$. We now list several consequences of this assumption that are direct to verify. For each $\mathbf{A} \in \mathscr{A}$ and each $\mathbf{X} \in \mathscr{X}$, the homset $\mathscr{A}(\mathbf{A}, \underline{\mathbf{M}})$ determines a closed substructure $D(\mathbf{A})$ of $\underset{\sim}{\mathbf{M}}^A$ and the homset $\mathscr{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ determines a subalgebra $E(X)$ of $\underline{\mathbf{M}}^X$. Thus we have $D\colon \mathscr{A} \to \mathscr{X}$ and $E\colon \mathscr{X} \to \mathscr{A}$. Moreover, if $f\colon \mathbf{A} \to \mathbf{B}$ in $\mathscr{A}$ and $\varphi\colon \mathbf{X} \to \mathbf{Y}$ in $\mathscr{X}$, then we define $D(f)\colon D(\mathbf{B}) \to D(\mathbf{A})$ by $D(f)(u) = u \circ f$ and $E(\varphi)\colon E(\mathbf{Y}) \to E(\mathbf{X})$ by $E(\varphi)(\sigma) = \sigma \circ \varphi$. The result is that D and E are faithful contravariant functors between $\mathscr{X}$ and $\mathscr{A}$. Consider now the compositions $ED\colon \mathscr{A} \to \mathscr{A}$ and $DE\colon \mathscr{X} \to \mathscr{X}$. For $\mathbf{A} \in \mathscr{A}$ and $\mathbf{X} \in \mathscr{X}$ we define the *evaluation maps*

$$e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A}) \quad \text{and} \quad \varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$$

by $e_{\mathbf{A}}(a)(u) = u(a)$ and $\varepsilon_{\mathbf{X}}(x)(\sigma) = \sigma(x)$. It is easy to check that both $e_{\mathbf{A}}$ and $\varepsilon_{\mathbf{X}}$ are always embeddings. For a more detailed discussion of these consequences of choosing $\underset{\sim}{\mathbf{M}}$ to be algebraic over $\underline{\mathbf{M}}$, see the first section of Clark and Davey [5].

The situation we have presented becomes a useful tool to describe the algebras in $\mathscr{A}$ if each evaluation map $e_{\mathbf{A}}$ is onto, that is, is an isomorphism. When this is the case for each $\mathbf{A} \in \mathscr{A}$ we say that $\underset{\sim}{\mathbf{M}}$ *yields a (natural) duality on* $\mathscr{A}$. If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathscr{A}$, then D turns out to be full as a contravariant functor, and each member $\mathbf{A}$ of $\mathscr{A}$ is represented as the algebra of all continuous homomorphisms from its dual space $D(\mathbf{A})$ into $\underset{\sim}{\mathbf{M}}$.

Given structures $\underset{\sim}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}'$ with common carrier M, let $\mathscr{X} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$ and let $\mathscr{X}' = \mathbb{I}\mathbb{S}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}'$. We say that $\underset{\sim}{\mathbf{M}}'$ *generates* $\underset{\sim}{\mathbf{M}}$ if $\mathscr{X}'(\mathbf{X}', \underset{\sim}{\mathbf{M}}') \subseteq \mathscr{X}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ whenever $\mathbf{X} \leq \underset{\sim}{\mathbf{M}}^S$ and $\mathbf{X}' \leq (\underset{\sim}{\mathbf{M}}')^S$ have the same carrier $X \subseteq M^S$. (A slightly weaker condition given in [5] is actually sufficient.) A simple and frequently needed consequence of the definition of duality is the following lemma.

$\underset{\sim}{\mathbf{M}}$-SHIFT DUALITY LEMMA 1.1. (Clark and Davey [5]) *If* $\underset{\sim}{\mathbf{M}}'$ *generates* $\underset{\sim}{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathscr{A}$, *then* $\underset{\sim}{\mathbf{M}}'$ *also yields a duality on* $\mathscr{A}$.

The next step is to find practically verifiable conditions under which a particular choice of $\underset{\sim}{\mathbf{M}}$ will yield a duality on $\mathscr{A}$. The structure $\underset{\sim}{\mathbf{M}}$ is *injective* in the category $\mathscr{X}$ if, for every morphism $\alpha: \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ and embedding $\varphi: \mathbf{X} \to \mathbf{Y}$ in $\mathscr{X}$, there is a morphism $\beta: \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$ such that $\beta \circ \varphi = \alpha$. This notion plays a central role in this study.

FIRST DUALITY THEOREM 1.2. (Davey and Werner [13]) *The following are equivalent*:

(a) $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathscr{A}$;

(b) *for all* $A \in \mathscr{A}$, *every morphism* $\alpha: D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ *extends to an* A*-ary term function* $\tau: M^A \to M$;

(c) *the following two conditions hold –*

(INJ) $\underset{\sim}{\mathbf{M}}$ *is injective with respect to those embeddings in* $\mathscr{X}$ *which are of the form* $D(u): D(\mathbf{B}) \to D(\mathbf{A})$ *where* $u: \mathbf{A} \to \mathbf{B}$ *is a surjective homomorphism, that is, for each morphism* $\alpha: D(\mathbf{B}) \to \underset{\sim}{\mathbf{M}}$ *there exists a morphism* $\beta: D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ *such that* $\beta \circ D(u) = \alpha$,

(CLO) *for each* $n \in \mathbb{N}$, *every morphism* $\tau: \underset{\sim}{\mathbf{M}}^n \to \underset{\sim}{\mathbf{M}}$ *is an* n*-ary term function on* $\underline{\mathbf{M}}$.

Because $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, every term function of $\underline{\mathbf{M}}$ is a morphism in $\mathscr{X}$. The condition (CLO) adds the converse: duality requires that the term functions on $\underline{\mathbf{M}}$ must be exactly the morphisms from powers of $\underset{\sim}{\mathbf{M}}$ into $\underset{\sim}{\mathbf{M}}$. Thus (CLO) says precisely that the structure on $\underset{\sim}{\mathbf{M}}$ determines the clone of term functions on $\underline{\mathbf{M}}$. Given that M is finite, it is easy to see that (CLO) implies the following stronger version (see Davey and Werner [13, 1.8(1)]):

(CLO)$^\infty$ *for each nonempty set* S, *the morphisms from* $\underset{\sim}{\mathbf{M}}^S$ *to* $\underset{\sim}{\mathbf{M}}$ *are precisely the* S*-ary term functions on* $\underline{\mathbf{M}}$.

Note that (INJ), which is a special case of the injectivity of $\underset{\sim}{\mathbf{M}}$ in $\mathscr{X}$, has been proven in every example to date by actually demonstrating that $\underset{\sim}{\mathbf{M}}$ is injective in $\mathscr{X}$. For the finite members of $\mathscr{X}$ we notice that (INJ) and (CLO) follow from the stronger but simpler *interpolation condition*:

(IC) *for each* $n \in \mathbb{N}$ *and each substructure* $\mathbf{X}$ *of* $\underset{\sim}{\mathbf{M}}^n$, *every morphism* $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ *extends to a term function* $\tau\colon M^n \to M$ *of the algebra* $\underline{\mathbf{M}}$.

The value of a duality is vastly strengthened if we have not only that each $e_{\mathbf{A}}$ is an isomorphism for $\mathbf{A} \in \mathscr{A}$, but also that each $\varepsilon_{\mathbf{X}}$ is an isomorphism for $\mathbf{X} \in \mathscr{X}$. When this is the case we say that $\underset{\sim}{\mathbf{M}}$ *yields a full duality on* $\mathscr{A}$. If $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathscr{A}$, then both D and E are full and faithful equivalences between the categories $\mathscr{A}$ and $\mathscr{X}$.

A special role will be played by the (possibly empty) set K of elements of M which determine one element subalgebras of $\underline{\mathbf{M}}$. If $\underset{\sim}{\mathbf{M}}$ is algebraic over $\underline{\mathbf{M}}$, then K determines a substructure $\mathbf{K} \leq \underset{\sim}{\mathbf{M}}$. Under a full duality we can say much more about $\mathbf{K}$.

LEMMA 1.3. *Assume that* $\underset{\sim}{\mathbf{M}}$ *yields a full duality on* $\mathscr{A}$.

(a) $\mathbf{K}$ *and* $\underline{\mathbf{1}}$ *are dual to one another*: $E(\mathbf{K}) \cong \underline{\mathbf{1}} \cong \mathbf{K}$.

(b) $\mathbf{K}$ *is the substructure of* $\underset{\sim}{\mathbf{M}}$ *generated by its distinguished elements.*

(c) *For every* $\mathbf{X} \in \mathscr{X}$ *there is a unique embedding of* $\mathbf{K}$ *into* $\mathbf{X}$.

(d) $\mathbf{K}$ *is an initial object* (*free object on the empty set*) *in* $\mathscr{X}$ *while* $\underline{\mathbf{1}}$ *is a final object in* $\mathscr{A}$.

(e) $\underset{\sim}{\varnothing} \in \mathscr{X}$ *if and only if* $\mathbf{K} = \underset{\sim}{\varnothing}$ *if and only if* $\underset{\sim}{\mathbf{M}}$ *has no nullary operations if and only if* $\mathscr{A}^+ \neq \mathscr{A}$.

Proof. (a) $D(\underline{\mathbf{1}}) \cong \mathbf{K}$ since the embeddings of $\underline{\mathbf{1}}$ into $\underline{\mathbf{M}}$ correspond exactly to the elements of K. Since we have a duality, $E(\mathbf{K}) \cong ED(\underline{\mathbf{1}}) \cong \underline{\mathbf{1}}$, that is, the only morphism from $\mathbf{K}$ into $\underset{\sim}{\mathbf{M}}$ is the inclusion. (b) Let $\mathbf{C}$ denote the (possibly empty) substructure of $\underset{\sim}{\mathbf{M}}$ generated by the distinguished elements of $\underset{\sim}{\mathbf{M}}$. Then, clearly, $E(\mathbf{C}) \cong \underline{\mathbf{1}} \cong E(\mathbf{K})$ and, since the duality is full, $\mathbf{C} \cong DE(\mathbf{C}) \cong DE(\mathbf{K}) \cong \mathbf{K}$. From the definition of $\mathbf{C}$, this isomorphism must be the identity, that is, $\mathbf{C} = \mathbf{K}$. Now (c), (d) and (e) follow immediately from (b). □

In view of item (b) above, in order to achieve full or strong duality, we are obliged to include a generating set worth of members of K as distinguished elements of $\underset{\sim}{\mathbf{M}}$. For $\mathbf{X} \in \mathscr{X}$ we will denote by $\mathbf{K_X}$ the image of $\mathbf{K}$ under the unique embedding of (c) above. If $\mathbf{A} \in \mathscr{A}$ and $a \in K$, we take $a_{D(\mathbf{A})}$ to be the constant map from A to a. Then

$$\mathbf{K}_{D(\mathbf{A})} = \{a_{D(\mathbf{A})} \mid a \in K\}.$$

Our next problem is to determine exactly when a duality is full. If $\underset{\sim}{\mathbf{M}}$ does yield a duality on $\mathscr{A}$, then it turns out that D and the restriction $E \restriction \mathscr{X}'$ are equivalences between $\mathscr{A}$ and the full subcategory $\mathscr{X}' = \mathbb{I}D(\mathscr{A}) \subseteq \mathscr{A}$, and $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathscr{A}$ if and only if $\mathscr{X}' = \mathscr{X}$. While it is in general difficult to know if a given structure $\mathbf{X}$ is in $\mathscr{X}'$, there are two alternate descriptions of $\mathscr{X}'$ which are often useful. Let $S \neq \varnothing$ and let $X \subseteq M^S$. Then X is said to be *hom-closed* if it is closed under all pointwise applications of homomorphisms of subalgebras of arbitrary powers of $\underline{\mathbf{M}}$ into $\underline{\mathbf{M}}$. Alternatively, X is said to be *term-closed* if it is an intersection of equalizers of S-ary term functions on $\underline{\mathbf{M}}$, that is, of morphisms from $\underset{\sim}{\mathbf{M}}^S$ into $\underset{\sim}{\mathbf{M}}$. These notions prove to be equivalent: X is term-closed if and only if X is hom-closed. These choices of X determine the dual category.

THEOREM 1.4. (Clark and Krauss [7, 2.26]) Let $\mathbf{X} \in \mathscr{X}$. *Then* $\mathbf{X} \in \mathscr{X}' = \mathbb{I}D(\mathscr{A})$ *if and only if* $\mathbf{X}$ *is isomorphic to a hom-closed* (= *term-closed*) *substructure of* $\underset{\sim}{\mathbf{M}}^S$ *for some nonempty set* S.

To prove that $\underset{\sim}{\mathbf{M}}$ yields a full duality, we must show that every closed substructure of a power of $\underset{\sim}{\mathbf{M}}$ is isomorphic to a hom-closed substructure of a power of $\underset{\sim}{\mathbf{M}}$. Now it turns out that the only way anyone has every succeeded in doing this is to prove an apparently stronger fact: every closed substructure of a power of $\underset{\sim}{\mathbf{M}}$ is, as it stands, hom-closed. When this is the case we say that $\underset{\sim}{\mathbf{M}}$ *yields a strong* (and therefore also full) *duality on* $\mathscr{A}$.

If every closed substructure of a power of $\underset{\sim}{\mathbf{M}}$ is to be hom-closed, then, in particular, every closed substructure of a power of $\underset{\sim}{\mathbf{M}}$ is closed under every algebraic partial operation on M. The strategy for making a duality strong is now clear. By adding algebraic partial operations to $\underset{\sim}{\mathbf{M}}$ we will not, according to the $\underset{\sim}{\mathbf{M}}$-Shift Duality Lemma, jeopardize our duality, but we will help to force closed substructure of powers of $\underset{\sim}{\mathbf{M}}$ to be hom-closed. If we are lucky, closure under some (finitely many) finitary algebraic partial operations will insure closure under all arbitrary algebraic partial operations, that is, hom-closure. When we can succeed in finding such a collection of finitary algebraic partial operations to add to $\mathbf{M}$, then we will have a strong duality.

We now quote several important results from Clark and Davey [5] that will be applied in this paper. The first provides a purely category-theoretic characterization of strong duality.

FIRST STRONG DUALITY THEOREM 1.5. *The structure* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$ *if and only if* $\underset{\sim}{\mathbf{M}}$ *yields a full duality on* $\mathscr{A}$ *and is injective in* $\mathscr{X}$.

We say that $\underset{\sim}{\mathbf{M}}$ is a *total structure* if it has no proper partial operations, that is, if $H = \varnothing$. We say that $\underset{\sim}{\mathbf{M}}$ satisfies the *Finite Term Closure* condition (FTC) if every substructure of a finite power of $\underset{\sim}{\mathbf{M}}$ is term-closed.

SECOND STRONG DUALITY THEOREM 1.6. (abridged) *Assume that* $\underset{\sim}{\mathbf{M}}$ *is a total structure that yields a duality on* $\mathscr{A}$. *Then* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$ *if and only if* (FTC) *holds.*

THIRD STRONG DUALITY THEOREM 1.7. (abridged) *Assume that* $\underset{\sim}{\mathbf{M}}$ *is a total structure and that* R *is finite. Then* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$ *if and only if* (IC) *and* (FTC) *hold.*

We are now ready to state the two culminating theorems of Clark and Davey [5] from which all presently known strong dualities can be obtained. For $k \geq 2$, a $(k+1)$-ary term τ is called a *near-unanimity term* for $\underline{\mathbf{M}}$ if $\tau(x, x, \ldots, y, x, \ldots, x) = x$ whenever all but perhaps one argument is equal to x. We will use the fact that the presence of a near-unanimity term for $\underline{\mathbf{M}}$ implies, among other things, that the variety generated by $\underline{\mathbf{M}}$ is congruence distributive. Given a finite algebra $\underline{\mathbf{M}}$ we define the integer $\mathrm{Irr}(\underline{\mathbf{M}})$ as the smallest number n such that the zero congruence of $\mathbf{A}$ is the meet of n meet irreducible congruences for each subalgebra $\mathbf{A}$ of $\underline{\mathbf{M}}$. In particular, $\mathrm{Irr}(\underline{\mathbf{M}}) = 1$ if and only if every subalgebra of $\underline{\mathbf{M}}$ is subdirectly irreducible.

NU-STRONG-DUALITY THEOREM 1.8. *Let* $k \geq 2$ *and assume that* $\underline{\mathbf{M}}$ *has a* $(k+1)$*-ary near-unanimity term. If the structure on* $\underset{\sim}{\mathbf{M}}$ *generates all subalgebras of* $\underline{\mathbf{M}}^k$ *and the clone of* $\underset{\sim}{\mathbf{M}}$ *includes all* n*-ary partial operations on* $\underline{\mathbf{M}}$ *for* $n \leq \mathrm{Irr}(\underline{\mathbf{M}})$, *then* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$.

TWO-FOR-ONE STRONG DUALITY THEOREM 1.9. (abridged) *Let* $\underline{\mathbf{M}} = \langle M; F\rangle$ *and assume that* $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathscr{T}\rangle$ *is a total algebra. Define* $\underline{\mathbf{M}}' = \langle M; G\rangle$ *and* $\mathscr{A}' = \mathbb{ISP}\underline{\mathbf{M}}'$, *and define* $\underset{\sim}{\mathbf{M}}' = \langle M; F, \mathscr{T}\rangle$ *and* $\mathscr{X}' = \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}'$. *Then the following are equivalent:*

(a) (IC) *and* (FTC) *hold with respect to* $\underset{\sim}{\mathbf{M}}$;
(b) (IC) *and* (FTC) *hold with respect to* $\underset{\sim}{\mathbf{M}}'$;
(d) (IC) *holds with respect to both* $\underset{\sim}{\mathbf{M}}$ *and* $\underset{\sim}{\mathbf{M}}'$;
(e) $\underset{\sim}{\mathbf{M}}'$ *and* $\underset{\sim}{\mathbf{M}}'$ *yield strong dualities on* $\mathscr{A}$ *and* $\mathscr{A}'$ *respectively*;
(f) $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$.

We conclude with a simple but useful analog of the $\underset{\sim}{\mathbf{M}}$-Shift Duality Lemma.

$\underset{\sim}{\mathbf{M}}$-SHIFT STRONG DUALITY LEMMA 1.10. *Assume that the structure* $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathscr{T} \rangle$ *yields a strong duality on* $\mathscr{A}$. *Then* $\underset{\sim}{\mathbf{M}}'$ *will also yield a strong duality on* $\mathscr{A}$ *if it is obtained from* $\underset{\sim}{\mathbf{M}}$ *by*:
(a) *enlarging* G, H *or* R, *or*
(b) *deleting members of* G *or* H *which can be obtained as compositions of the remaining members of* G *and* H *and the projection mappings.*
Moreover, if $\underset{\sim}{\mathbf{M}}'$ *yields a duality on* $\mathscr{A}$ *and is obtained from* $\underset{\sim}{\mathbf{M}}$ *by*
(c) *deleting members of* R, *or*
(d) *deleting members of* H *which have an extension in* G *or* H,
then $\underset{\sim}{\mathbf{M}}'$ *will also yield a strong duality on* $\mathscr{A}$.

This ends our mini crash-course in duality theory. Again, we refer the reader to Clark and Davey [5] or to Davey [9] for more detail. Now that we understand what makes a *natural duality*, what makes a *full duality* and what makes a *strong duality*, we are ready to address the most important question: What makes a *good duality*?

2. Total structures and injectivity

Perhaps the first restriction on the type of $\mathscr{X}$ that most of us would like to see would be the elimination of the set H of partial operations. The theory of strong dualities, as developed in Clark and Krauss [7] and in Clark and Davey [5], makes it clear that partial operations play an inherent role in the description of the dual category and can therefore generally be expected to appear in its type. Moreover there are many very tractable dualities which utilize proper partial operations. But there was a number of central results of Clark and Davey [5] for producing strong dualities which apply only when $\underset{\sim}{\mathbf{M}}$ is a *total structure*, that is, when H is empty, and this is a context in which many readers are likely to feel more comfortable. These results culminated in the Two-for-One Strong Duality Theorem which gives strong dualities for the variety generated by a finite abelian group, vector space, set or semilattice via some category of total algebras. Given that there exists some choice of $\underset{\sim}{\mathbf{M}}$ which yields a strong duality on $\mathscr{A}$, our first goal in this section is to determine exactly when and how $\underset{\sim}{\mathbf{M}}$ can be replaced by a total structure which yields a strong duality on $\mathscr{A}$. We will begin by looking for methods of constructing, from a given choice of $\underset{\sim}{\mathbf{M}}$ which yields a strong duality on $\mathscr{A}$, new structures $\underset{\sim}{\mathbf{M}}'$ which also yield a strong duality on $\mathscr{A}$.

Let $\underset{\sim}{\mathbf{M}}'$ be a structure over the set M having the discrete topology but an otherwise arbitrary type. We say that $\underset{\sim}{\mathbf{M}}'$ *dominates* $\underset{\sim}{\mathbf{M}}$ if, for every nonempty set S, a closed subset X of M^S determines a substructure of $\underset{\sim}{\mathbf{M}}^S$ whenever it determines a substructure of $(\underset{\sim}{\mathbf{M}}')^S$, and, moreover, a map between these substructures is a morphism with respect to $\underset{\sim}{\mathbf{M}}$ whenever it is a morphism with respect to $\underset{\sim}{\mathbf{M}}'$. For example, $\underset{\sim}{\mathbf{M}}'$ dominates $\underset{\sim}{\mathbf{M}}$ if $\underset{\sim}{\mathbf{M}}'$ is obtained from $\underset{\sim}{\mathbf{M}}$ by adding (partial) operations or relations. Notice that if both $\underset{\sim}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}'$ are total algebras (having neither partial operations nor relations), then the condition on substructure implies the condition on morphisms by the usual argument: f is a morphism if and only if its graph is a closed substructure.

Our next lemma, which is an immediate consequence of the definitions, says that we will not lose a duality or a strong duality by strengthening the structure on $\underset{\sim}{\mathbf{M}}$. (It is worth observing that the corresponding statement is not an immediate consequence of the definition of 'full duality'.)

LEMMA 2.1. *Assume that* $\underset{\sim}{\mathbf{M}}'$ *dominates* $\underset{\sim}{\mathbf{M}}$.

(a) *If* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathscr{A}$, *then* $\underset{\sim}{\mathbf{M}}'$ *also yields a duality on* $\mathscr{A}$.

(b) *If* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$, *then* $\underset{\sim}{\mathbf{M}}'$ *also yields a strong duality on* $\mathscr{A}$.

In view of Lemma 2.1 we would like to find ways to construct useful choices of $\underset{\sim}{\mathbf{M}}'$ which dominate $\underset{\sim}{\mathbf{M}}$. The next lemma is also routine to verify.

LEMMA 2.2. *Let* $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathscr{T}\rangle$ *and assume that the partial n-ary operation* $h \in H$ *extends to a homomorphism* $g\colon \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$. *Let* $\underset{\sim}{\mathbf{M}}' = \langle M; G \cup \{g\}, H\backslash\{h\}, R \cup \{\mathrm{dom}(h)\}, \mathscr{T}\rangle$. *Then* $\underset{\sim}{\mathbf{M}}'$ *dominates* $\underset{\sim}{\mathbf{M}}$.

COROLLARY 2.3. *Assume that* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality* (*a duality*) *on* $\mathscr{A}$ *and that each partial n-ary operation* $h \in H$ *extends to a homomorphism* $g\colon \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$. *Let* $\underset{\sim}{\mathbf{M}}'$ *be obtained from* $\underset{\sim}{\mathbf{M}}$ *by replacing each partial operation h with the total operation g and the n-ary relation* dom(h). *Then* $\underset{\sim}{\mathbf{M}}'$ *is a total structure which yields a strong duality* (*a duality*) *on* $\mathscr{A}$.

Proof. Apply Lemmas 2.1 and 2.2. □

When does $\underset{\sim}{\mathbf{M}}$ have the property that each of its partial operations extends to a total algebraic operation on $\underline{\mathbf{M}}$? Notice that this property is simply a special case of the *injectivity* of $\underline{\mathbf{M}}$ in $\mathscr{A}$. Thus we can use Corollary 2.3 to eliminate partial operations whenever $\underline{\mathbf{M}}$ is injective in $\mathscr{A}$.

This observation sounds promising since we know that $\underset{\sim}{\mathbf{M}}$ must be injective in $\mathcal{X}$ if $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathcal{A}$. The corresponding statement about $\underline{\mathbf{M}}$ is false, however, is shown by Theorem 5.2(c) of Clark and Davey [5]. There a strong duality was obtained for the quasi-variety generated by the Heyting algebra 4-chain in which H includes a partial algebraic operation that does not extend to a total algebraic operation, thereby violating the injectivity of $\underline{\mathbf{M}}$ in $\mathcal{A}$. Partial operations of $\underset{\sim}{\mathbf{M}}$ obstruct the injectivity of $\underline{\mathbf{M}}$ in $\mathcal{A}$ because of an insidious problem they can cause: in the presence of partial operations, the image of a structure under a morphism is not in general a substructure! This observation bears directly on the question of injectivity, as shown by the following fact.

INJECTIVITY LEMMA 2.4. (Clark and Davey [5]) *Assume that* $\underset{\sim}{\mathbf{M}}$ yields a full duality on $\mathcal{A}$.

(a) *If* $\underline{\mathbf{M}}$ *is injective in* $\mathcal{A}$, *then* $\underset{\sim}{\mathbf{M}}$ *is injective in* $\mathcal{X}$.

(b) If $\underset{\sim}{\mathbf{M}}$ *is injective in* $\mathcal{X}$, *then* $\underline{\mathbf{M}}$ *is injective in* $\mathcal{A}$ *provided that images under morphisms in* $\mathcal{X}$ *are always substructures.*

The proof of part (b) is identical to that of (a), but the proof of (a) uses the fact that image of an algebra in $\mathcal{A}$ under a homomorphism is always a subalgebra.

The Injectivity Lemma tells us that, if we want to know when partial operations can be eliminated, we must determine when images in $\mathcal{X}$ are substructures. This problem, in turn, brings us back to the notion of domination. We say that structures $\underset{\sim}{\mathbf{M}}$ and $\underset{\sim}{\mathbf{M}}'$, having the same carrier but arbitrary types, are *structurally equivalent* if each dominates the other. Now if $\underset{\sim}{\mathbf{M}}$ is structurally equivalent to a total structure, then images in $\mathcal{X}$ must certainly be substructures.

Thus $\underset{\sim}{\mathbf{M}}$ can be replaced by a total structure which also yields a strong duality on $\mathcal{A}$ whenever it is structurally equivalent to some total structure $\underset{\sim}{\mathbf{M}}'$. In that case Lemma 2.1 tells us that $\underset{\sim}{\mathbf{M}}'$ itself is a total structure which also yields a strong duality on $\mathcal{A}$. While structural equivalence may seem like a lot to ask for, our last lemma shows that it is actually necessary.

LEMMA 2.5. *Assume that* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathcal{A}$. *Then, for any structure* $\underset{\sim}{\mathbf{M}}'$, *the following are equivalent*:

(a) $\underset{\sim}{\mathbf{M}}'$ *yields a strong duality on* $\mathcal{A}$;

(b) $\underset{\sim}{\mathbf{M}}'$ *is structurally equivalent to* $\underset{\sim}{\mathbf{M}}$.

Proof. That (b) implies (a) follows from Lemma 2.1. To prove that (a) implies (b) we note that the same sets determine substructures of powers of $\underset{\sim}{\mathbf{M}}$ and of $\underset{\sim}{\mathbf{M}}'$, namely the hom-closed sets. Assume that $X \subseteq M^S$ and $Y \subseteq M^T$ are hom-closed and that $\varphi\colon X \to Y$ preserves the structure of $\underset{\sim}{\mathbf{M}}$. Since products in $\mathcal{X}$ are cartesian, to see that φ preserves the structure of $\underset{\sim}{\mathbf{M}}'$, we must check that $\pi_t \circ \varphi\colon X \to M$ preserves the structure of $\underset{\sim}{\mathbf{M}}'$ for each $t \in T$. Since $\underset{\sim}{\mathbf{M}}$ is injective in $\mathscr{X}$, we conclude that $\pi_t \circ \varphi$ extends to a morphism $\psi\colon \underset{\sim}{\mathbf{M}}^S \to \underset{\sim}{\mathbf{M}}$. Using (CLO)$^\infty$ we see that ψ must be an S-ary term function on $\underline{\mathbf{M}}$ since $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathscr{A}$. But term functions preserve all algebraic structure on M; in particular they preserve the structure of $\underset{\sim}{\mathbf{M}}'$. □

This result now brings us back to the beginning: if $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathscr{A}$, then the only time some total structure $\underset{\sim}{\mathbf{M}}'$ also yields a strong duality on $\mathscr{A}$ is when it is structurally equivalent to $\underset{\sim}{\mathbf{M}}$! The facts we have established collectively show that an array of conditions are equivalent under a strong duality. We gather these conditions in the next theorem which tells exactly when we can eliminate partial operations from a given strong duality.

TOTAL STRUCTURE THEOREM 2.6. *If $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathscr{A}$, then the following are equivalent*:

(a) *some total structure $\underset{\sim}{\mathbf{M}}'$ yields a strong duality on $\mathscr{A}$*;
(b) *for each natural number n, every n-ary partial operation $h \in H$ in the structure on $\underset{\sim}{\mathbf{M}}$ extends to a homomorphism $g\colon \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$*;
(c) *$\underline{\mathbf{M}}$ is injective in $\mathscr{A}$*;
(d) *the image of every morphism in $\mathscr{X}$ is a closed substructure*;
(e) *$\underset{\sim}{\mathbf{M}}$ is structurally equivalent to a total structure.*

In particular, $\underset{\sim}{\mathbf{M}}'$ in (a) *may be obtained from $\underset{\sim}{\mathbf{M}}$ by deleting H and using* (b) *to replace each partial operation $h \in H$ with an extension g to a total operation in G and with its domain* dom(h) *as a relation in R.*

Proof. Observe that (a) implies (e) by Lemma 2.5, (e) trivially implies (d), and (c) follows from (d) by the Injectivity Lemma. Condition (b) is simply the special case of (c) applied to inclusion maps of finite structures, and (b) implies (a) by Corollary 2.3. □

A sharpened local version of the implication (d) ⇒ (b) of the Total Structure Theorem depends only on the preduality yielded by $\underset{\sim}{\mathbf{M}}$ and further pinpoints the role of partial operations. Consider simultaneously all algebraic partial operations h having a fixed domain $\mathbf{A} \le \underline{\mathbf{M}}^n$. Let $u\colon \mathbf{A} \to \underline{\mathbf{M}}^n$ be the inclusion map and consider its dual, the restriction map $D(u)\colon D(\underline{\mathbf{M}}^n) \to D(\mathbf{A})$. The image U of $D(\underline{\mathbf{M}}^n)$ under $D(u)$ consists exactly of those $h\colon \mathbf{A} \to \underline{\mathbf{M}}$ which can be extended to a homomorphism $g\colon \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$. We will discover which h are in U by exploiting their schizophrenic nature as both morphisms in $\mathscr{A}$ and as partial operations on the objects in $\mathscr{X}$.

PROPOSITION 2.7. *Let n be a positive integer, let* $\mathbf{A}$ *be a subalgebra of* $\underline{\mathbf{M}}^n$ *and let* $h: \mathbf{A} \to \underline{\mathbf{M}}$ *be a homomorphism. Then h extends to a homomorphism* $g: \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$ *if and only if the image U of* $D(\underline{\mathbf{M}}^n)$ *under the morphism* $D(u)$ *is closed under h.*

Proof. The assertion tht U is closed under h may be stated in full as follows: If $x_1, \ldots, x_n \in D(\underline{\mathbf{M}}^n)$ and $\langle x_1(a), \ldots, x_n(a) \rangle \in A$ for each $a \in A$, then there is a homomorphism $k: \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$ such that, for all $a \in A$ we have $k(a) = h(x_1(a), \ldots, x_n(a))$. Now this is clearly the case if h itself extends to a homomorphism $g: \underline{\mathbf{M}}^n \to \mathbf{M}$:

Conversely, suppose U is closed under h and let x_i be the projection π_i for $i = 1, \ldots, n$. Trivially, $\langle \pi_1(a), \ldots, \pi_n(a) \rangle = a \in A$ for each $a \in A$! Consequently we have $g(a) = h(\pi_1(a), \ldots, \pi_n(a)) = h(a)$ for each $a \in A$. □

EXAMPLE 2.8. Here is a simple but useful application of the Total Structure Theorem. If $\underline{\mathbf{M}}$ is a quasi-primal algebra, then we can apply the NU-Strong-Duality Theorem to obtain the strong duality of Davey and Werner [13] where G is the automorphism group of $\underline{\mathbf{M}}$ together with the one element subalgebras of $\underline{\mathbf{M}}$ as distinguished elements, H is the set of internal isomorphisms of $\underline{\mathbf{M}}$ and $R = \varnothing$. Now assume further that $\underline{\mathbf{M}}$ is a *semi-primal algebra*, that is, it has no proper automorphisms or isomorphisms between non-trivial subalgebras. (A number of interesting examples of semi-primal algebras are described in Werner [18].) The resulting choice of $\underset{\sim}{\mathbf{M}}$ is rather ungainly, with a distinguished element for each trivial subalgebra and with its collection of partial identity maps corresponding to each of the subalgebras of $\underline{\mathbf{M}}$! A much more tractable strong duality is obtained from the Total Structure Theorem by eliminating all of the identity maps and taking R to consist of all proper subalgebras of $\underline{\mathbf{M}}$ viewed as unary relations. The resulting representations of the algebras in $\mathscr{A}$ are known as *filtered Boolean products*. (See Arens and Kaplansky [2] or Burris and Clark [3].)

Even better than obtaining a strong duality from a total structure ($H = \varnothing$) is obtaining a strong duality from a total algebra ($R = H = \varnothing$). The Two-for-One Strong Duality Theorem gives us a list of equivalent conditions, any one of which can be checked to determine if *a particular* total algebra $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathscr{T} \rangle$ yields a strong duality on $\mathscr{A}$. We now give our second description of good dualities which tells when *some* choice of a total algebra yields a strong duality on $\mathscr{A}$.

TOTAL ALGEBRA THEOREM 2.9. *The following are equivalent*:

(a) *there is a total algebra* $\underset{\sim}{\mathbf{M}}$ *which yields a strong duality on* $\mathscr{A}$;

(b) *there is a structure* $\underset{\sim}{\mathbf{M}}$ *which yields a strong duality on* $\mathscr{A}$ *such that*

(i) *the image of every morphism in* $\mathscr{X}$ *is a substructure,*

(ii) *every one-to-one morphism in* $\mathscr{X}$ *is an embedding*;

(c) $\underline{\mathbf{M}}$ *is injective in* $\mathscr{A}$ *and epis are surjective in* $\mathscr{A}$;

(d) $\underline{\mathbf{M}}$ *is injective in* $\mathscr{A}$ *and, for all* $\mathbf{A} \in \mathscr{A}$, *every subalgebra of* $\mathbf{A}$ *is an intersection of equalizers (of homomorphisms from* $\mathbf{A}$ *into* $\underline{\mathbf{M}}$);

(d)′ $\underline{\mathbf{M}}$ *is injective in* $\mathscr{A}$ *and, for all* $\mathbf{A} \in \mathscr{A}$, *every subalgebra of* $\mathbf{A}$ *is an equalizer of a pair of homomorphisms from* $\mathbf{A}$ *into* $\mathbf{B}$ *for some* $\mathbf{B} \in \mathscr{A}$;

(e) $\underline{\mathbf{M}}$ *is injective in* $\mathscr{A}$ *and, for all* $n \in \mathbb{N}$, *every subalgebra of* $\underline{\mathbf{M}}^n$ *is an intersection of equalizers (of homomorphisms from* $\underline{\mathbf{M}}^n$ *into* $\underline{\mathbf{M}}$).

(e)′ $\underline{\mathbf{M}}$ *is injective in* $\mathscr{A}$ *and, for all* $n \in \mathbb{N}$, *every subalgebra of* $\underline{\mathbf{M}}^n$ *is an equalizer of a pair of homomorphisms from* $\underline{\mathbf{M}}^n$ *into* $\mathbf{B}$ *for some finite* $\mathbf{B} \in \mathscr{A}$.

Moreover, if $\underset{\sim}{\mathbf{M}}$ *satisfies* (b), *then* $\underset{\sim}{\mathbf{M}}$ *is structurally equivalent to a total algebra* $\underset{\sim}{\mathbf{M}}'$ *which may be obtained from* $\underset{\sim}{\mathbf{M}}$ *by first replacing each partial map h by its domain,* dom *h, and any total extension, g, of the map h and then replacing each n-ary relation r by a finite family,* G_r, *of total functions,* $g\colon M^n \to M$, *such that r is an intersection of equalizers of pairs of functions from* G_r.

Proof. That (a) implies (b) is trivial.

Assume that (b) holds. By the Total Structure Theorem, (b)(i) implies tht $\underline{\mathbf{M}}$ is injective in $\mathscr{A}$. If $f\colon \mathbf{A} \to \mathbf{B}$ is an epi in $\mathscr{A}$, then $D(f)\colon D(\mathbf{B}) \to D(\mathbf{A})$ is one-to-one and hence, by (b)(ii), is an embedding. Since $\underset{\sim}{\mathbf{M}}$ is injective in $\mathscr{X}$, the map $ED(f)$ is surjective, whence $f = e_{\mathbf{B}}^{-1} \circ ED(f) \circ e_{\mathbf{A}}$ is surjective. Thus (b) implies (c).

Let $\mathbf{B} \leq \mathbf{A} \in \mathscr{A}$ and define

$$C := \bigcap \{\mathrm{Eq}(x_1, x_2) \mid x_1, x_2 \in \mathscr{A}(\mathbf{A}, \underline{\mathbf{M}}) \text{ and } x_1\restriction_{\mathbf{B}} = x_2\restriction_{\mathbf{B}}\}$$

We claim that, provided $\underline{\mathbf{M}}$ is injective in $\mathscr{A}$, the inclusion of $\mathbf{B}$ into $\mathbf{C}$ is an epi. Let $y_1, y_2 \in \mathscr{A}(\mathbf{C}, \underline{\mathbf{M}})$ with $y_1\restriction_{\mathbf{B}} = y_2\restriction_{\mathbf{B}}$ and suppose that $y_1(c) \neq y_2(c)$ for some $c \in C \backslash B$. Since $\underline{\mathbf{M}}$ is injective, there exist $x_1, x_2 \in \mathscr{A}(\mathbf{A}, \underline{\mathbf{M}})$ with $x_i\restriction_C = y_i$. Thus $x_1\restriction_B = x_2\restriction_B$ and $c \notin \mathrm{Eq}(x_1, x_2)$, whence $c \notin C$, a contradiction. Thus the inclusion of $\mathbf{B}$ into $\mathbf{C}$ is an epi. Consequently, (c) implies (d).

Again, (d) implies (e) is trivial.

Assume that $\underline{\mathbf{M}}$ is injective in $\mathscr{A}$ and that, for all $n \in \mathbb{N}$, every subalgebra of $\underline{\mathbf{M}}^n$ is an intersection of equalizers of homomorphisms from $\underline{\mathbf{M}}^n$ into $\underline{\mathbf{M}}$. We shall prove that

$$\underset{\sim}{\mathbf{M}} := \langle M; \bigcup \{\mathscr{A}(\underline{\mathbf{M}}^n, \underline{\mathbf{M}}) \mid n \in \mathbb{N}\}, \mathscr{T}\rangle$$

yields a strong duality on $\mathscr{A}$. If $\underline{\mathbf{M}} = \langle M; F\rangle$ then we define

$$\underline{\mathbf{M}}' := \langle M; \bigcup \{\mathscr{A}(\underline{\mathbf{M}}^n, \underline{\mathbf{M}}) \mid n \in \mathbb{N}\}, \rangle \quad \text{and} \quad \underset{\sim}{\mathbf{M}}' := \langle M; F, \mathscr{T}\rangle.$$

Since $\underline{\mathbf{M}}$ is injective it follows at once that (IC) holds with respect to $\underset{\sim}{\mathbf{M}}'$ and $\underline{\mathbf{M}}'$.

The statement that every subalgebra of each finite power of $\underline{\mathbf{M}}$ is an intersection of equalizers is precisely (FTC) with respect to $\underset{\sim}{\mathbf{M}}'$. Thus, by the Two-for-One Duality Theorem, $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathscr{A}$. Hence, (e) implies (a).

The equivalence of (d) and (d)′ (respectively, (e) and (e)′) follows from the fact that $\mathscr{A} = \mathbb{ISP}\underline{\mathbf{M}}$.

To prove the 'Moreover', assume that $\underline{\mathbf{M}}$ satisfies (b). By (e) we can construct the total algebra $\underline{\mathbf{M}}'$, and by the $\underset{\sim}{\mathbf{M}}$-Shift Strong Duality Lemma we know that $\underset{\sim}{\mathbf{M}}'$ also yields a strong duality on $\mathscr{A}$. Lemma 2.5 now tells us that $\underset{\sim}{\mathbf{M}}'$ is structurally equivalent to $\underset{\sim}{\mathbf{M}}$. □

3. Unary structures, coproducts and near unanimity

Apart from the examples cited at the beginning of the previous section that arise from the Two-for-One Strong Duality Theorem, every specific strong duality that has proven useful to date has a dual category that is tractable in a quite different way: all operations and partial operations are at most unary. This restriction can result in an especially nice dual category. For example, it means that substructures of any structure in $\mathscr{X}$ are closed under unions and consequently form a *distributive* lattice. The NU-Strong-Duality-Theorem tells us that a strong duality meeting this restriction exists in case $\underset{\sim}{\mathbf{M}}$ has a near-unanimity term and every subalgebra of $\underline{\mathbf{M}}$ is subdirectly irreducible. In this section we will first prove the Unary Structure Theorem which asserts that these conditions are all equivalent. Then we will add the hypothesis that $\underset{\sim}{\mathbf{M}}$ is a total structure as well. The Unary Total Structure Theorem will tell us that this is equivalent to asserting that coproducts in $\mathscr{X}$ are always given as unions of the factors.

We shall need to quote two recent results, the first of which uses duality to describe congruences. Let $\mathbf{A} = E(\mathbf{X})$. For each closed substructure $\mathbf{Z} \subseteq \mathbf{X}$ we define the *projection congruence*

$$\theta_Z := \{(\alpha, \beta) \mid \alpha(z) = \beta(z) \text{ for all } z \in Z\}$$

on $\mathbf{A}$. Recall that we have insured that the substructures of a member of $\mathscr{X}$ will be a lattice by allowing the empty structure into $\mathscr{X}$.

PROPOSITION 3.1. (Davey and Priestley [11, Proposition 1.12]) *Assume that $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathscr{A}$ and that $\mathbf{A} = E(\mathbf{X})$ where $\mathbf{X} \in \mathscr{X}$. Then the lattice of closed substructures of $\mathbf{X}$ is dually isomorphic to the lattice*

$$\mathrm{Con}_{\mathscr{A}}(\mathbf{A}) := \{\theta \in \mathrm{Con}\,\mathbf{A} \mid \mathbf{A}/\theta \in \mathscr{A}\}$$

of $\mathscr{A}$-congruences on $\mathbf{A}$ under the mapping $\mathbf{Z} \mapsto \theta_Z$.

An early observation of the general theory of natural dualities was that the presence of a near-unanimity term for $\underline{\mathbf{M}}$ was sufficient to insure that it admitted a duality. Recently a partial converse to this result was discovered. We say that $\mathscr{A}$ is *relatively congruence distributive* if, for each $\mathbf{A} \in \mathscr{A}$, the lattice $\mathrm{Con}_{\mathscr{A}}(\mathbf{A})$ is distributive.

THEOREM 3.2. (Davey, Heindorf and McKenzie [10]) *If $\mathscr{A}$ is relatively congruence distributive and admits a duality, then* $\mathbf{A}$ *has a near-unanimity term.*

LEMMA 3.3. *Let* $\mathbf{A}$ *be a subalgebra of* $\underline{\mathbf{M}}^S$, *let* $\varnothing \neq T \subseteq S$ *and let*

$$X_T := \{u\colon A \to M \mid \text{for all } a, b \in A, \quad \text{if } a \restriction T = b \restriction T, \text{ then } u(a) = u(b)\}$$

be the set of maps from A to M that depend only on T. Then X_T is a closed substructure of $\underset{\sim}{\mathbf{M}}^A$.

Proof. Let $h \in G \cup H$ and let $(x_1, \dots, x_n) \in \mathrm{dom}_{X_T}(h)$. Thus $(x_1, \dots, x_n) \in \mathrm{dom}_{\underset{\sim}{\mathbf{M}}^A}(h)$ and $x_1, \dots, x_n \in X_T$. Let $a, b \in A$ with $a \restriction T = b \restriction T$. Hence $x_i(a) = x_i(b)$ for all i, and consequently

$$\begin{aligned} h(x_1, \dots, x_n)(a) &= h(x_1(a), \dots, x_n(a)) \\ &= h(x_1(b), \dots, x_n(b)) \\ &= h(x_1, \dots, x_n)(b). \end{aligned}$$

Thus $h(x_1, \dots, x_n)$ depends only on T and is therefore in X_T. Hence X_T is a substructure of $\underset{\sim}{\mathbf{M}}^A$.

Let $u \in M^A \backslash X_T$. Thus there exist $a, b \in A$ with $a \restriction T = b \restriction T$ and $u(a) \neq u(b)$. Now the set of points in M^A which agree with u at a and b is an open set which contains u and is disjoint from X_T. Thus X_T is closed in M^A. □

We can now prove our next main result, whose equivalences we list beginning with the desired condition on the type of $\underset{\sim}{\mathbf{M}}$ ending with a finitely verifiable condition on $\underline{\mathbf{M}}$.

UNARY STRUCTURE THEOREM 3.4. *The following are equivalent*:

(a) *there is a structure $\underset{\sim}{\mathbf{M}}$ with each operation in $G \cup H$ at most unary which yields a strong duality on $\mathscr{A}$;*

(b) *there is a structure $\underset{\sim}{\mathbf{M}}$ such that the substructures of powers of $\underset{\sim}{\mathbf{M}}$ are closed under unions and $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathscr{A}$;*

(c) *the variety generated by $\underline{\mathbf{M}}$ is congruence distributive, every subalgebra of $\underline{\mathbf{M}}$ is subdirectly irreducible and $\mathscr{A}$ admits a duality;*

(d) $\underline{\mathbf{M}}$ *has a near-unanimity term and every subalgebra of* $\underline{\mathbf{M}}$ *is subdirectly irreducible.*

Moreover if these conditions hold, then there is a single choice of $\underset{\sim}{\mathbf{M}}$ *of finite type which satisfies* (a), (b) *and* (c) *simultaneously.*

Proof. The assertion (c) $\Rightarrow$ (d) follows from Theorem 3.2, and (d) $\Rightarrow$ (c) follows from the NU-Strong-Duality Theorem and the fact that near unanimity implies congruence distributivity. Again, (d) $\Rightarrow$ (a) follows from the NU-Strong-Duality Theorem and (a) trivially implies (b).

Now assume (b) to prove (d). For each $\mathbf{A} \in \mathscr{A}$ the lattice of $\mathscr{A}$-congruences together with the universal congruence is, by Proposition 3.1, dually isomorphic to the lattice of closed substructures of $D(\mathbf{A})$ under union and intersection, and this is a distributive lattice. Consequently $\mathscr{A}$ is relatively congruence distributive. From Theorem 3.2 we conclude that $\underline{\mathbf{M}}$ has a near-unanimity term.

In order to complete the proof of (b) $\Rightarrow$ (d), it remains to show that every subalgebra $\mathbf{Q}$ of $\underline{\mathbf{M}}$ is subdirectly irreducible. Let $\theta_1, \theta_2 \in \operatorname{Con} \mathbf{Q}$ such that $\theta_1 \cap \theta_2 = \mathbf{0}^Q$. We will show that either $\theta_1 = \mathbf{0}^Q$ or $\theta_2 = \mathbf{0}^Q$. Let $\mathbf{A} := \theta_1 \circ \theta_2$, a subalgebra of $\underline{\mathbf{M}}^2$. Since $\theta_1 \cap \theta_2 = \mathbf{0}^Q$, for all $(a, b) \in A$ there is a unique $c \in M$ such that $a\theta_1 c\theta_2 b$. Hence we may define a map $h: A \to M$ by $h(a, b) = c$. Since θ_1 and θ_2 are congruences, h is a homomorphism. Define

$$X_1 := \{u: A \to M \mid u \text{ depends only on the first coordinate}\}$$

$$X_2 := \{u: A \to M \mid u \text{ depends only on the second coordinate}\}.$$

By Lemma 3.3, the sets $\mathbf{X}_1$ and $\mathbf{X}_2$ are substructures of $\underset{\sim}{\mathbf{M}}^A$. By (b) we have that $\mathbf{X} = \mathbf{X}_1 \cup \mathbf{X}_2$ is also a substructure of $\underset{\sim}{\mathbf{M}}^A$.

From the definition of strong duality we know that $\mathbf{X}$ is hom-closed in $\underset{\sim}{\mathbf{M}}^A$. Let $\pi_i: A \to M$ be the projection for $i = 1, 2$. Then $\pi_1, \pi_2 \in X$ and clearly $(\pi_1, \pi_2) \in \operatorname{dom}_X(h)$ since

$$(\pi_1, \pi_2)(a, b) = (\pi_1(a, b), \pi_2(a, b)) = (a, b) \in A = \operatorname{dom}(h)$$

for all $(a, b) \in A$. Since $\mathbf{X}$ is hom-closed, $h = h(\pi_1, \pi_2) \in X$. Without loss of generality, assume that $h \in X_1$, that is, h depends only on its first coordinate. We claim that $\theta_1 = \mathbf{0}^Q$. Indeed,

$$\begin{aligned}(a, b) \in \theta_1 \quad &\Rightarrow \quad a\theta_1 a\theta_2 a \text{ and } a\theta_1 b\theta_2 b \\ &\Rightarrow \quad h(a, a) = a \text{ and } h(a, b) = b \\ &\Rightarrow \quad a = b\end{aligned}$$

since $h(a, a) = h(a, b)$. Hence $\mathbf{Q}$ is subdirectly irreducible, as required.

To verify the 'Moreover' statement, observe that under condition (d) the NU-Strong-Duality Theorem yields (a) with a finite type for $\underset{\sim}{\mathbf{M}}$, and that our proofs of (a) $\Rightarrow$ (b) and (b) $\Rightarrow$ (c) each retain the same choice of $\underset{\sim}{\mathbf{M}}$. Thus the nontrivial part of this assertion is the fact that (c) implies (d), which uses Theorem 3.2 to allow us to reduce any choice of $\underset{\sim}{\mathbf{M}}$ to one of finite type. □

The best natural dualities seem to be the ones that combine the restrictions of the two preceding theorems, as we see for example in the cases of the strong dualities for distributive lattices, Kleene algebras, de Morgan algebras, Stone and double Stone algebras, the varieties $\mathscr{B}_n$ of distributive p-algebras and the variety of rings generated by a finite field (see [13] or [5] for descriptions of these dualities and corresponding references to the literature) and for semi-primal algebras (Example 2.8). These dualities share yet another special property which contributes further toward our contention that they are 'good dualities'. This property concerns products.

In the category $\mathscr{A}$ direct products (which are concrete, i.e. formed pointwise) always exist and are denoted by

$$\mathbf{A} \xleftarrow[\pi_1]{} \mathbf{A} \times \mathbf{B} \xrightarrow[\pi_2]{} \mathbf{B},$$

where π_1 and π_2 are the associated surjective projections. Under a full duality (and therefore, by the First Strong Duality Theorem, also under a strong duality) direct products in $\mathscr{A}$ are always dual to coproducts in $\mathscr{X}$. In particular, *coproducts always exist in* $\mathscr{X}$ and will be denoted by

$$\mathbf{X} \xrightarrow[\sigma_1]{} \mathbf{X} * \mathbf{Y} \xleftarrow[\sigma_1]{} \mathbf{Y},$$

where σ_1 and σ_2 are the associated maps. Ready access to the coproduct $D(\mathbf{A}) * D(\mathbf{B})$ means a convenient representation for the direct product $\mathbf{A} \times \mathbf{B}$ in $\mathscr{A}$. Under the conditions of the Unary Structure Theorem coproducts in $\mathscr{X}$ have a very nice property which by itself has nothing to do with duality.

LEMMA 3.5. *Suppose that the coproduct*

$$\mathbf{X} \xrightarrow[\sigma_1]{} \mathbf{X} * \mathbf{Y} \xleftarrow[\sigma_2]{} \mathbf{Y}$$

of members $\mathbf{X}$ *and* $\mathbf{Y}$ *of* $\mathscr{X}$ *exists in* $\mathscr{X}$. *Then the morphisms* σ_1 *and* σ_2 *are both embeddings. If in addition every operation in* $G \cup H$ *is at most unary, then* $\mathbf{X} * \mathbf{Y} = \sigma_1(\mathbf{X}) \cup \sigma_1(\mathbf{Y})$ *is the union of the images.*

Proof. To see that σ_1 is an embedding, we must show that distinct elements, unrelated elements and elements outside of the domain of a partial operation are kept so under a morphism from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$. We use the fact that $\mathbf{X}$ can be embedded into a power of $\underset{\sim}{\mathbf{M}}$. As a result we can find, for example, a projection of $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$ that separates two arbitrary distinct elements. Since this projection factors through σ_1, we conclude that σ_1 must also separate those two elements. The other parts of the argument are similar.

Since σ_1 and σ_2 are embeddings, the images $\sigma_1(\mathbf{X})$ and $\sigma_2(\mathbf{Y})$ are both closed substructures of $\mathbf{X} * \mathbf{Y}$. Since the operations of $G \cup H$ are at most unary, their union $\mathbf{Z} = \sigma_1(\mathbf{X}) \cup \sigma_2(\mathbf{Y})$ is a closed substructure of $\mathbf{X} * \mathbf{Y}$. It follows directly from the definition of coproduct that $\mathbf{Z}$ is also a coproduct of $\mathbf{X}$ and $\mathbf{Y}$.

To see that $\mathbf{Z}$ must be all of $\mathbf{X} * \mathbf{Y}$, consider the commuting diagram of Figure 2. Let $\eta: \mathbf{Z} \to \mathbf{X} * \mathbf{Y}$ be the inclusion and let $\alpha: \mathbf{X} * \mathbf{Y} \to \mathbf{Z}$ complete the lower part of the diagram. Since $\eta \circ \alpha$ is the unique map that completes the outer diagram, it must be the identity on $\mathbf{X} * \mathbf{Y}$. Thus η is surjective. □

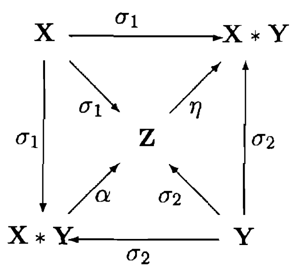

Figure 2. Coproducts as unions.

If coproducts in $\mathscr{X}$ satisfy $\mathbf{X} * \mathbf{Y} = \sigma_1(\mathbf{X}) \cup \sigma_2(\mathbf{Y})$, we shall say that *coproducts in $\mathscr{X}$ are given by union*. It follows from Lemma 3.5 that the conditions of the Unary Structure Theorem imply that coproducts in $\mathscr{X}$ are given by union. If $\underset{\sim}{\mathbf{M}}$ is a total structure as well, we will show that the converse also holds. Recall that $\underline{\mathbf{M}}$ is called *self injective* if every homomorphism from a subalgebra of $\underline{\mathbf{M}}$ into $\underline{\mathbf{M}}$ extends to an endomorphism of $\underline{\mathbf{M}}$.

UNARY TOTAL STRUCTURE THEOREM 3.6. *The following are equivalent*:

(a) *there is a total structure $\underset{\sim}{\mathbf{M}}$ with each operation in G at most unary* (*and R finite*) *which yields a strong duality on $\mathscr{A}$*;

(b) *there is a total structure $\underset{\sim}{\mathbf{M}}$* (*of finite type*) *such that substructures of powers of $\underset{\sim}{\mathbf{M}}$ are closed under unions and $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathscr{A}$*;

(c) *there is a total structure* $\underset{\sim}{\mathbf{M}}$ (*of finite type*) *such that coproducts in* $\mathscr{X}$ *are given by union and* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$;

(d) $\underline{\mathbf{M}}$ *has a near-unanimity term, every subalgebra of* $\underline{\mathbf{M}}$ *is subdirectly irreducible and* $\underline{\mathbf{M}}$ *is injective in* $\mathscr{A}$;

(e) $\underline{\mathbf{M}}$ *has a near-unanimity term, every subalgebra of* $\underline{\mathbf{M}}$ *is subdirectly irreducible and* $\underline{\mathbf{M}}$ *is self injective.*

Proof. The equivalence of (a), (b) and (d) follows from the corresponding equivalence in the Unary Structure Theorem together with the Total Structure Theorem, (a) implies (c) by Lemma 3.5, and (d) trivially implies (e). We shall complete the proof by showing that (e) implies (d) and that (c) implies (b).

Assume (e). By the NU-Strong-Duality Theorem we have a structure $\underset{\sim}{\mathbf{M}}$ yielding a strong duality on $\mathscr{A}$ in which each operation and partial operation is at most unary. Applying the fact that $\underline{\mathbf{M}}$ is self injective together with Total Structure Theorem ('In particular, . . .'), we can replace $\underset{\sim}{\mathbf{M}}$ with a total structure $\underset{\sim}{\mathbf{M}}'$ which yields a strong duality on $\mathscr{A}$. Now (d) follows from the implication (a) $\Rightarrow$ (c) of the Total Structure Theorem.

Finally assume (c) and let $\mathbf{X}$ and $\mathbf{Y}$ be substructures of $\underset{\sim}{\mathbf{M}}^S$. By the definition of coproduct, there is a morphism $\varphi\colon \mathbf{X} * \mathbf{Y} \to \underset{\sim}{\mathbf{M}}^S$ such that $\varphi \circ \sigma_1$ and $\varphi \circ \sigma_2$ are the inclusion maps of $\mathbf{X}$ and $\mathbf{Y}$ into $\underset{\sim}{\mathbf{M}}^S$, respectively. By part (d) of the Total Structure Theorem, the image of φ, namely $\mathbf{X} \cup \mathbf{Y}$, is a substructure of $\underset{\sim}{\mathbf{M}}^S$. □

4. Better coproducts and logarithmicity

As we observed in the last section, coproducts always exist in $\mathscr{X}$ under a strong duality and are dual to direct products in $\mathscr{A}$. We would therefore ask that a 'good duality' should provide a simple means of constructing the coproduct of any two structures in $\mathscr{X}$, thereby giving us a nice representation of a finite product of algebras from representations of the factors. For example, the coproduct of two Boolean spaces is just their disjoint union. Under the conditions of the Unary Total Structure Theorem coproducts are always obtainable as the union of copies of the factors. Unfortunately, this in itself is not enough information to allow us to construct a coproduct from its factors for two reasons:

(a) we need to know what the intersection of the factors is to be;

(b) we need to know how to define nonunary relations on the union.

The case of Boolean algebras is easy because the intersection is empty and there are no nonunary relations in the dual category. In this section we will determine exactly when there is a simple resolution to (b). But first we show that there is always a simple answer to (a). (See Section 1 for a discussion of the structures **K** and $\mathbf{K_X}$.)

LEMMA 4.1. *If* $\underset{\sim}{\mathbf{M}}$ *yields a full duality on* $\mathscr{A}$, *then for any coproduct*

$$\mathbf{X} \xrightarrow[\sigma_1]{} \mathbf{X} * \mathbf{Y} \xleftarrow[\sigma_2]{} \mathbf{Y}$$

in $\mathscr{X}$, *we have* $\sigma_1(\mathbf{X}) \cap \sigma_2(\mathbf{Y}) = \mathbf{K}_{\mathbf{X} * \mathbf{Y}}$.

Proof. We represent **X** and **Y** as $D(\mathbf{A})$ and $D(\mathbf{B})$, represent $\mathbf{X} * \mathbf{Y}$ as $D(\mathbf{A} \times \mathbf{B})$, and σ_1 and σ_2 as $D(\pi_1)$ and $D(\pi_2)$. Then for $u \in D(\mathbf{A} \times \mathbf{B})$ we observe that

$$u \in D(\pi_1)(D(\mathbf{A})) \cap D(\pi_2)(D(\mathbf{B}))$$

$$\Leftrightarrow \quad u = f \circ \pi_1 = g \circ \pi_2 \qquad \text{for some } f \in D(\mathbf{A}) \text{ and } g \in D(\mathbf{B})$$

$$\Leftrightarrow \quad u(c, d) = f(c) = g(d) \qquad \text{for all } c \in A, d \in B$$

$$\Leftrightarrow \quad u = a_{D(\mathbf{A} \times \mathbf{B})} \qquad \text{for some } a \in K.$$

□

Assume the conditions of the Unary Structure Theorem: $\underset{\sim}{\mathbf{M}}$ yields a strong duality on $\mathscr{A}$ and each member of $G \cup H$ is at most unary. How can we obtain a constructive description of a coproduct of two structures **X** and **Y** in $\mathscr{X}$? In order to do this we shall define the *direct union* of **X** and **Y** to be the structure whose base set is the union of their base sets with $\mathbf{K_X}$ and $\mathbf{K_Y}$ amalgamated, whose topology is the unique topology which makes the inclusion maps embeddings, and whose operations, partial operations and relations are each the union of the corresponding operations, partial operations and relations of **X** and **Y**. If the coproduct of every pair of structures in $\mathscr{X}$ is their direct union, we shall say that *coproducts in* $\mathscr{X}$ *are given by direct union.* In general, of course, we would not expect $\mathscr{X}$ even to be closed under direct unions. The following observation is independent of duality.

PROPOSITION 4.2. *If* $\mathscr{X}$ *is closed under direct unions, then coproducts in* $\mathscr{X}$ *are given by direct union.*

Proof. Assume that $\mathscr{X}$ is closed under direct unions. Let $\mathbf{X}, \mathbf{Y} \in \mathscr{X}$ and let

$$\mathbf{X} \xrightarrow[\gamma_1]{} \mathbf{Z} \xleftarrow[\gamma_2]{} \mathbf{Y}$$

be embeddings into their direct union **Z**. To show that **Z** is their coproduct, consider any $\mathbf{W} \in \mathscr{X}$ and morphisms $\rho_1 \colon \mathbf{X} \to \mathbf{W}$ and $\rho_2 \colon \mathbf{Y} \to \mathbf{W}$. Let $\tau \colon Z \to W$ be the unique map such that $\tau \circ \rho_i = \gamma_i$ for $i = 1, 2$. From the definition of direct union it is now easy to check that τ is a morphism in $\mathscr{X}$. □

If $\underset{\sim}{\mathbf{M}}$ yields a strong duality and coproducts in $\mathscr{X}$ are given by direct union, then we may think of the duality as relating products in $\mathscr{A}$ to 'sums' in $\mathscr{X}$. Under these conditions we will say that $\underset{\sim}{\mathbf{M}}$ yields a *logarithmic duality* on $\mathscr{A}$. For example, if each member of $G \cup H \cup R$ is at most unary, then a strong duality will be logarithmic. Unfortunately most interesting instances of the Unary Structure Theorem require binary relations, often in the form of order relations. Consider, for example, the duality between distributive lattices $\langle D; \vee, \wedge \rangle$ and bounded Priestley spaces $\langle X; 0, 1, \leq \rangle$. In this case $K = \{0, 1\}$ and it is immediate to check that $\mathscr{X}$ is closed under direct unions. Applying Proposition 4.2 we see that the duality is logarithmic. Rather surprisingly we discover, by examining them case by case, that every other familiar strong duality in which each member of $G \cup H$ is at most unary also turns out to be logarithmic!

This phenomenon is explained by the following theorem, which gives a simple finitary test for logarithmeticity. It shows that all of the strong dualities given in the previous section as illustrations of the Unary Total Structure Theorem as well as the strong duality of Werner [17] for certain quasi-varieties generated by weakly associative lattices are logarithmic. The key to logarithmeticity is a simple property of the relations in R. We will say that an algebraic relation $r \leq \underline{\mathbf{M}}^n$ (for $n \geq 2$) *avoids binary products* if each binary projection

$$\pi_{ij}(r) := \{(\pi_i(a), \pi_j(a)) \in M^2 \mid a \in r\},$$

for $1 \leq i < j \leq n$, contains no product $A \times B$ of nontrivial subalgebras of $\underline{\mathbf{M}}$.

LOGARITHMIC DUALITY THEOREM 4.3. *Assume the conditions of the Unary Structure Theorem: $\underline{\mathbf{M}}$ has a $(k+1)$-ary near-unanimity term and has only subdirectly irreducible subalgebras. Then the following are equivalent:*

(a) *there is an $\underset{\sim}{\mathbf{M}}$ which yields a logarithmic duality on $\mathscr{A}$;*
(b) *there is an $\underset{\sim}{\mathbf{M}}$ which yields a strong duality on $\mathscr{A}$ where the coproduct in $\mathscr{X}$ of every pair of finite structures is given by direct union;*
(c) *there is an $\underset{\sim}{\mathbf{M}}$ which yields a strong duality on $\mathscr{A}$ where for all $\mathbf{A}, \mathbf{B} \leq \underline{\mathbf{M}}$, the coproduct of $D(\mathbf{A})$ and $D(\mathbf{B})$ is given by direct union;*
(d) *there is an $\underset{\sim}{\mathbf{M}}$ which yields a strong duality on $\mathscr{A}$ where for all $n \geq 2$, every n-ary relation $r \in R$ avoids binary products;*
(e) *the set of algebraic relations $r \leq \underline{\mathbf{M}}^k$ which avoid binary products generate all k-ary algebraic relations on $\underline{\mathbf{M}}$.*

Moreover, any choice of $\underline{\mathbf{M}}$ satisfying one of (a), (b), (c) *or* (d) *also satisfies the others.*

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are trivial. Assume that (c) holds. Taking the same choice of $\underset{\sim}{\mathbf{M}}$ we will prove (d). Let $r \in R$ be n-ary for some $n \geq 2$

and assume that there are subalgebras **A** and **B** of $\underline{\mathbf{M}}$ such that $A \times B \subseteq \pi_{ij}(r)$ for some i and j. We will show that either **A** or **B** must be trivial. We represent the coproduct of $D(\mathbf{A})$ and $D(\mathbf{B})$ as

$$D(\mathbf{A}) \xrightarrow[\sigma_1]{} D(\mathbf{A} \times \mathbf{B}) \xleftarrow[\sigma_2]{} D(\mathbf{B}),$$

where $\sigma_1(x) = x \circ \pi_1$ and $\sigma_2(y) = y \circ \pi_2$ for $x \in D(\mathbf{A})$ and $y \in D(\mathbf{B})$. Now choose $x: \mathbf{A} \to \underline{\mathbf{M}}$ and $y: \mathbf{B} \to \underline{\mathbf{M}}$ to be the inclusions. Then

$$\begin{aligned} & A \times B \subseteq \pi_{ij}(r) \\ & \quad \Rightarrow \quad (x(a), y(b)) \in \pi_{ij}(r) \qquad \text{for all } (a, b) \in A \times B \\ & \quad \Rightarrow \quad (x \circ \pi_1(a, b), y \circ \pi_2(a, b)) \in \pi_{ij}(r) \qquad \text{for all } (a, b) \in A \times B \\ & \quad \Rightarrow \quad (x \circ \pi_1, y \circ \pi_2) \in \pi_{ij}(r) \qquad \text{on } D(\mathbf{A} \times \mathbf{B}) \\ & \quad \Rightarrow \quad (\sigma_1(x), \sigma_2(y)) \in \pi_{ij}(r) \qquad \text{on } D(\mathbf{A} \times \mathbf{B}). \end{aligned}$$

Since $D(\mathbf{A} \times \mathbf{B})$ is the direct union of $\sigma_1(D(\mathbf{A}))$ and $\sigma_2(D(\mathbf{B}))$ it follows that either $x \circ \pi_1 = \sigma_1(x) \in K_{D(\mathbf{A} \times \mathbf{B})}$ or $y \circ \pi_2 = \sigma_2(y) \in K_{D(\mathbf{A} \times \mathbf{B})}$. Thus one of the inclusions x or y maps to a single element of **K**, and therefore either **A** or **B** is trivial. This proves that (d) holds.

Now assume that (d) holds in order to prove (a). Let $\mathbf{A}, \mathbf{B} \in \mathscr{A}$, let $r \in R$ be n-ary for some $n \geq 2$ and let $z_1, \ldots, z_n \in D(\mathbf{A} \times \mathbf{B})$ with $(z_1, \ldots, z_n) \in r$. Assume that there exist i and j such that $z_i \in \sigma_1(D(\mathbf{A}))$ and $z_j \in \sigma_2(D(\mathbf{B}))$. We must show that either $z_i \in K_{D(\mathbf{A} \times \mathbf{B})}$ or $z_j \in K_{D(\mathbf{A} \times \mathbf{B})}$.

Let $z_i = x \circ \pi_i$ for some $x \in D(\mathbf{A})$ and let $z_j = y \circ \pi_j$ for some $y \in D(\mathbf{B})$. Then

$$\begin{aligned} & (z_1, \ldots, z_n) \in r \\ & \quad \Rightarrow \quad (z_i(a, b), z_j(a, b)) \in \pi_{ij}(r) \qquad \text{for all } (a, b) \in A \times B \\ & \quad \Rightarrow \quad (x \circ \pi_i(a, b), y \circ \pi_j(a, b)) \in \pi_{ij}(r) \qquad \text{for all } (a, b) \in A \times B \\ & \quad \Rightarrow \quad (x(a), y(b)) \in \pi_{ij}(r) \qquad \text{for all } (a, b) \in A \times B \\ & \quad \Rightarrow \quad x(A) \times y(B) \subseteq \pi_{ij}(r). \end{aligned}$$

Thus, by (*d*), either $x(A)$ is trivial, and hence $z_i = x \circ \pi_i \in K_{D(\mathbf{A} \times \mathbf{B})}$, or $y(B)$ is trivial, in which case $z_j = y \circ \pi_j \in K_{D(\mathbf{A} \times \mathbf{B})}$.

Finally, (d) and (e) are equivalent by the NU-Strong Duality Theorem. □

According to Lemma 4.1, images of factors in a coproduct in $\mathscr{X}$ will be disjoint exactly when $\mathbf{K} = \underset{\sim}{\varnothing}$. If coproducts in $\mathscr{X}$ are given by unions and $\mathbf{K} = \underset{\sim}{\varnothing}$, we say that *coproducts in $\mathscr{X}$ are given by disjoint union.* Combining Lemma 1.3 with the Unary Total Structure Theorem we immediately obtain a full description of the conditions under which this happens.

UNARY TOTAL STRUCTURE WITHOUT CONSTANTS THEOREM 4.4. *The following are equivalent*:

(a) *there is a total structure* $\underset{\sim}{\mathbf{M}}$ *with each operation in* G *at most unary* (*and* R *finite*), *such that the monoid of maps generated by* G *contains no constant maps, which yields a strong duality on* $\mathscr{A}$;

(b) *there is a total structure* $\underset{\sim}{\mathbf{M}}$ (*of finite type*) *such that substructures of powers of* $\underset{\sim}{\mathbf{M}}$ *are closed under unions, the clone of* $\underset{\sim}{\mathbf{M}}$ *contains no constant maps, and* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$;

(c) *there is a total structure* $\underset{\sim}{\mathbf{M}}$ (*of finite type*) *such that coproducts in* $\mathscr{X}$ *are given by disjoint union and* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$;

(d) $\underline{\mathbf{M}}$ *has a near-unanimity term, every subalgebra of* $\underline{\mathbf{M}}$ *is subdirectly irreducible,* $\underline{\mathbf{M}}$ *has no one-element subalgebras and* $\underline{\mathbf{M}}$ *is injective in* $\mathscr{A}$;

(e) $\underline{\mathbf{M}}$ *has a near-unanimity term, every subalgebra of* $\underline{\mathbf{M}}$ *is subdirectly irreducible,* $\underline{\mathbf{M}}$ *has no one element subalgebras and* $\underline{\mathbf{M}}$ *is self injective.*

5. Exclusively unary structures and arithmeticity

In this last section we will look for strong dualities in which the operations, partial operations and relations of $\underset{\sim}{\mathbf{M}}$ are at most unary. In this case we say that the members of $\mathscr{X}$ are *exclusively unary structures*. In particular such dualities will be logarithmic.

With few exceptions the strong dualities found to date have resulted from the NU-Strong Duality Theorem. In the presence of a $(k+1)$-ary near-unanimity term on $\underline{\mathbf{M}}$, it returns a strong duality with k-ary relations. Since k must be at least 2, this never leaves us less than binary relations. To eliminate binary relations we need something new. The key turns out to be the presence of a term τ on $\underline{\mathbf{M}}$ satisfying

$$\tau(x, y, y) = \tau(x, y, x) = \tau(y, y, x) = x.$$

Such a term is called a *Pixley arithmeticity term*, and its presence witnesses the fact that the algebras in the variety generated by $\underline{\mathbf{M}}$ all have a distributive lattice of permuting congruences. Simultaneously it gives us the 3-ary near-unanimity term $\tau(x, \tau(x, y, z), z)$ which leads to a strong duality with at most binary relations. While congruence distributivity is subsumed under near-unanimity, congruence permutability is new. It will allow us to use Fleischer's Theorem [14] to recognize the binary relations of $\underset{\sim}{\mathbf{M}}$ (which are subagebras of $\underline{\mathbf{M}} \times \underline{\mathbf{M}}$) and under the right conditions replace them with partial unary operations.

LEMMA 5.1. *Suppose congruences on subalgebras of* $\underline{\mathbf{M}} \times \underline{\mathbf{M}}$ *permute and that each nontrivial homomorphic image of a subalgebra of* $\underline{\mathbf{M}}$ *is in* $\mathscr{A}$. *Then any operation on* M *which preserves all algebraic unary partial operations of* $\underline{\mathbf{M}}$ *also preserves all algebraic binary relations of* $\underline{\mathbf{M}}$.

Proof. Let $r \leq \underline{\mathbf{M}} \times \underline{\mathbf{M}}$ be binary. Since congruences on $\underline{\mathbf{M}} \times \underline{\mathbf{M}}$ permute, we can use Fleischer's characterization of its subalgebras (Fleischer [14]). Let $\mathbf{P}$ and $\mathbf{Q}$ be subalgebras of $\underline{\mathbf{M}}$, let $\mathbf{C}$ be an algebra, and let

$$u: \mathbf{P} \to \mathbf{C} \quad \text{and} \quad v: \mathbf{Q} \to \mathbf{C}$$

be surjections so that $r = u \circ v^{-1}$ is the set of $(p, q) \in P \times Q$ such that $u(p) = v(q)$. Since $\mathbf{C} \in \mathbb{ISP}\underline{\mathbf{M}}$, there are homomorphisms $h_1, h_2, \ldots, h_m: \mathbf{C} \to \underline{\mathbf{M}}$ that separate $\mathbf{C}$.

Now let $f: M^n \to M$ be an n-ary operation that preserves all unary partial operations on $\underline{\mathbf{M}}$, and let $(a_1, b_1), \ldots, (a_n, b_n) \in r$. Then for each j we observe that $h_j \circ u: \mathbf{P} \to \underline{\mathbf{M}}$ and $h_j \circ v: \mathbf{Q} \to \underline{\mathbf{M}}$ are preserved by f. Thus $uf(a_1, \ldots, a_n)$ and $uf(b_1, \ldots, b_n)$ are in $\mathbf{C}$. To see that they are equal we check that they have the same value under each h_j.

$$\begin{aligned} h_j(uf(a_1, \ldots, a_n)) &= (h_j \circ u)(f(a_1, \ldots, a_n)) \\ &= f(h_j(u(a_1)), \ldots, h_j(u(a_n))) \\ &= f(h_j(v(b_1)), \ldots, h_j(v(b_n))) \\ &= (h_j \circ v)(f(b_1, \ldots, b_n)) \\ &= h_j(vf(b_1, \ldots, b_n)). \end{aligned}$$ □

Conversely, we find that strong duality by exclusively unary structures actually implies that congruences on members of $\mathscr{A}$ are arithmetical.

LEMMA 5.2. *Suppose that each member of* $G \cup H$ *is at most unary and each member of* R *is either unary or is a congruence on a subalgebra of* $\underline{\mathbf{M}}$. *If* $\underset{\sim}{\mathbf{M}}$ *yields a duality on* $\mathscr{A}$ *and* $\underline{\mathbf{M}}$ *is injective in* $\mathscr{X}$, *then* $\underline{\mathbf{M}}$ *has a Pixley arithmeticity term.*

Proof. Let $X \subseteq M^3$ consist of all triples (a, b, b), (a, b, a) and (b, b, a) where $a, b \in M$. Then X determines a substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^3$. Define $\sigma: X \to M$ by $\sigma(a, b, b) = \sigma(a, b, a) = \sigma(b, b, a) = a$. Clearly σ preserves every distinguished element, every unary (partial) operation and every unary relation. It is also straightforward to check that σ preserves every equivalence relation on a subalgebra of $\underline{\mathbf{M}}$. Consequently σ is a morphism which, by injectivity, extends to a morphism from $\underset{\sim}{\mathbf{M}}^3$ into $\underset{\sim}{\mathbf{M}}$. By (CLO), this morphism is a term function of $\underline{\mathbf{M}}$. □

UNARY PARTIAL ALGEBRA THEOREM 5.3. *Assume that every nontrivial homomorphic image of a subalgebra of* $\underline{\mathbf{M}}$ *is in* $\mathscr{A}$. *Then the following are equivalent*:

(a) *there is an* $\underset{\sim}{\mathbf{M}}$ *which yields a strong (necessarily logarithmic) duality on* $\mathscr{A}$ *where each member of* $G \cup H$ *is at most unary and* R *is empty*;

(b) *there is an* $\underset{\sim}{\mathbf{M}}$ *which yields a strong duality on* $\mathscr{A}$ *where each member of* $G \cup H$ *is at most unary and each member of* R *is either unary or is a congruence*;

(c) $\underline{\mathbf{M}}$ *generates an arithmetical variety and has only subdirectly irreducible subalgebras.*

Proof. Trivially (a) implies (b), and (b) implies (c) by Lemma 5.2 and the Unary Structure Theorem. It remains to prove that (c) implies (a). Assuming (c), the NU-Strong-Duality Theorem says that $\underset{\sim}{\mathbf{M}}'$ will yield a strong duality on $\mathscr{A}$ if we take G to be all endomorphisms of $\underline{\mathbf{M}}$, take H to be all homomorphisms between proper subalgebras of $\underline{\mathbf{M}}$, and take R to be all subalgebras of $\underline{\mathbf{M}}^2$.

Now let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, \mathscr{T} \rangle$ be obtained from $\underset{\sim}{\mathbf{M}}'$ by eliminating R. Jonsson's Theorem tells us that every homomorphic image of a subalgebra of $\underline{\mathbf{M}}$ is in $\mathscr{A}$. By Lemma 5.1 and the $\underline{\mathbf{M}}$-Shift Duality Lemma, $\underset{\sim}{\mathbf{M}}$ also yields a duality on $\mathscr{A}$, and by the $\underset{\sim}{\mathbf{M}}$-Shift Strong Duality Lemma, part (c), it yields a strong duality on $\mathscr{A}$. □

Strong dualities by unary partial algebras are obtained by Davey and Werner [13] for the quasi-variety generated by any quasi-primal algebra and by Clark and Davey [5] for the quasi-variety generated by the Heyting algebra 4-chain.

Of course the hypothesis of the Unary Partial Algebra Theorem is often not available, for in general the quasi-variety $\mathscr{A}$ is not closed under homomorphic images and is therefore much smaller than the variety generated by $\underline{\mathbf{M}}$. In many cases of natural dualities, however, $\mathscr{A}$ is a variety. This is due to a rather surprising fact: strong duality by exclusively unary *total* structures actually gives us closure of $\mathscr{A}$ under homomorphisms!

The proof of this result is a fine example of dualities at work. Our argument depends upon viewing a congruence θ on an algebra $\mathbf{A}$ as a subalgebra of $\mathbf{A} \times \mathbf{A}$, and then considering the dual $D(\theta)$ and the coproduct $D(\mathbf{A}) * D(\mathbf{A})$.

THEOREM 5.4. *Assume that an exclusively unary total structure* $\underset{\sim}{\mathbf{M}}$ *yields a strong duality on* $\mathscr{A}$. *Then* $\mathscr{A}$ *is closed under homomorphic images and hence is the variety generated by* $\underline{\mathbf{M}}$.

Proof. Represent $\mathbf{A} \in \mathscr{A}$ via the bijection $e_{\mathbf{A}}: \mathbf{A} \to ED(\mathbf{A}) \leq \underline{\mathbf{M}}^{D(A)}$. Under a strong duality Proposition 3.1 asserts that every $\mathscr{A}$-congruence on $\mathbf{A}$ is induced by the projection onto a closed substructure $\mathbf{Z} \leq D(\mathbf{A})$. Under a strong duality by an exclusively unary total structure, we will show that every congruence θ on $\mathbf{A}$ is induced by such a projection.

By the Unary Total Structure Theorem the coproduct

$$D(\mathbf{A}) \xrightarrow[D(\pi_1)]{} D(\mathbf{A}) * D(\mathbf{A}) \xleftarrow[D(\pi_2)]{} D(\mathbf{A})$$

is the union $(D(\mathbf{A}) \circ \pi_1) \cup (D(\mathbf{A}) \circ \pi_2)$ of two copies of $D(\mathbf{A})$. Let ρ_i be the restriction of π_i to θ. By the Total Structure Theorem, $\underline{\mathbf{M}}$ is injective in $\mathscr{A}$ and consequently

$$D(\theta) = (D(\mathbf{A}) \circ \rho_1) \cup (D(\mathbf{A}) \circ \rho_2).$$

Let $Z = \{x \in D(\mathbf{A}) \mid \theta \subseteq \ker x\}$ and let $\theta_Z = \cap\{\ker x \mid x \in Z\}$. For $a, b \in A$, notice that $(a, b) \in \theta_Z$ if and only if $x(a) = x(b)$ for all $x \in Z$, if and only if $e_{\mathbf{A}}(a)(x) = e_{\mathbf{A}}(b)(x)$ for all $x \in Z$, that is, $e_{\mathbf{A}}(a)$ and $e_{\mathbf{A}}(b)$ agree on Z. Thus θ_Z is just the congruence induced by the projection onto Z.

Clearly $\theta \subseteq \theta_Z$. We will show that $\theta = \theta_Z$. Let $(a, b) \in \theta_Z$ and let

$$\gamma\colon D(\theta) \to \underset{\sim}{\mathbf{M}} \qquad \text{where } \gamma(u \circ \rho_1) = u(a) \text{ and } \gamma(v \circ \rho_2) = v(b).$$

To see that γ is well-defined, assume that $u \circ \rho_1 = v \circ \rho_2$. Then, for any $c \in A$, we have $(c, c) \in \theta$ and therefore

$$u(c) = (u \circ \rho_1)(c, c) = (v \circ \rho_2)(c, c) = v(c).$$

Thus $u = v$. Now for every $(c, d) \in \theta$, we observe that

$$u(c) = (u \circ \rho_1)(c, d) = (v \circ \rho_2)(c, d) = v(d) = u(d).$$

This means tht $\theta \subseteq \ker u$ and, since $(a, b) \in \theta_Z$, that $u(a) = u(b) = v(b)$.

To see that γ is a continuous homomorphism it is sufficient, since $D(\theta)$ is an exclusively unary structure, to check that its restriction to each $D(\mathbf{A}) \circ \rho_i$ is a continuous homomorphism. For $u \in D(\mathbf{A})$ we have

$$(\gamma \circ D(\pi_1))(u) = \gamma(D(\pi_1)(u)) = \gamma(u \circ \rho_1) = u(a) = e_{\mathbf{A}}(a)(u).$$

Consequently on $D(\mathbf{A}) \circ \rho_1$ we see that γ agrees with the continuous homomorphism $e_{\mathbf{A}}(a) \circ D(\pi_1)^{-1}$. Similarly, γ agrees with $e_{\mathbf{A}}(b) \circ D(\pi_2)^{-1}$ on $D(\mathbf{A}) \circ \rho_2$.

Since $\gamma: D(\theta) \to \underset{\sim}{\mathbf{M}}$ is in $ED(\theta)$, we conclude from duality that there is a pair $(c, d) \in \theta$ such that $\gamma = e_\theta(c, d)$. Thus $u(c) = (u \circ \rho_1)(c, d) = \gamma(u \circ \rho_1) = u(a)$ for all $u \in D(\mathbf{A})$. Since $\mathbf{A} \in \mathscr{A}$, we conclude that $c = a$. Similarly, $d = b$ and $(a, b) = (c, d) \in \theta$, showing that $\theta = \theta_Z$ as claimed. Thus $\mathbf{A}/\theta$ is embedded in $\underline{\mathbf{M}}^Z$ and is therefore in $\mathscr{A}$. □

EXCLUSIVELY UNARY TOTAL STRUCTURE THEOREM 5.5. *The following are equivalent*:

(a) *there is an exclusively unary total structure* $\underset{\sim}{\mathbf{M}}$ *that yields a (necessarily logarithmic) strong duality on* $\mathscr{A}$;

(b) $\mathscr{A}$ *is an arithmetical variety in which* $\underline{\mathbf{M}}$ *is injective and has only subdirectly irreducible subalgebras*;

(c) $\underline{\mathbf{M}}$ *has a Pixley arithmeticity term, is self injective and has only subdirectly irreducible subalgebras, each homomorphic image of which is in* $\mathscr{A}$.

Proof. Self injectivity in (c) implies injectivity in (b) by the Unary Total Structure Theorem. Otherwise the equivalence of (b) and (c) follows from Pixley's Theorem and Jonsson's Theorem. Now assume (a). Then Theorem 5.4 assures us tht $\mathscr{A}$ is an arithmetical variety, $\underline{\mathbf{M}}$ is injective in $\mathscr{A}$ by the Total Structure Theorem and $\underline{\mathbf{M}}$ has only subdirectly irreducible subalgebras by the Unary Structure Theorem. This proves (b).

It remains to prove that (b) implies (a). Assuming (b), we obtain from the NU-Strong-Duality Theorem a choice of $\underset{\sim}{\mathbf{M}}$ which yields a strong duality on $\mathscr{A}$ such that each member of $G \cup H$ is at most unary and each member of R is at most binary. By Lemma 5.1 and the $\underset{\sim}{\mathbf{M}}$-Shift Strong Duality Lemma, each binary relation in R can be eliminated by adding unary (partial) operations in $G \cup H$. Since $\underline{\mathbf{M}}$ is self injective, we can use Corollary 2.3 to eliminate H by adding unary operations to G and unary relations to R. □

Strong duality by exclusively unary total structures is, of course, very restrictive. Example 2.8 shows that such strong duality exists for a semi-primal quasi-variety in which both G and H are empty. Conditions for a finite Heyting algebra to be strongly endodualizable (G is at most unary and H and R are both empty) are given in Clark and Davey [5, Theorem 5.1]. Among these are the Heyting algebra 2- and 3-chains and the Heyting algebra $2^2 \oplus 1$. We can now combine the above theorem with results of Clark and Davey [5] to give a finitely verifiable characterization of strong endodualizability in general.

UNARY TOTAL ALGEBRA THEOREM 5.6. *The following are equivalent*:

(a) $\underline{\mathbf{M}}$ *is strongly endodualizable*;

(b) $\underline{\mathbf{M}}$ *has a Pixley arithmeticity term, is self injective and each subalgebra* $\mathbf{Q}$ *of* $\underline{\mathbf{M}}$ *satisfies these conditions*:

(i) $\mathbf{Q}$ *is subdirectly irreducible*;
(ii) *each homomorphic image of* $\mathbf{Q}$ *is in* $\mathcal{A}$;
(iii) $\mathbf{Q}$ *is an intersection of equalizers of endomorphisms of* $\underline{\mathbf{M}}$.

Proof. Assuming (a), all but item (iii) of (b) follow from the Exclusively Unary Total Structure Theorem. To prove (iii) we apply two theorems of Clark and Davey [5]. By the Third Strong Duality Theorem, (IC) and (FTC) hold. Now let $\underset{\sim}{\mathbf{M}}'$ be obtained from $\underset{\sim}{\mathbf{M}}$ by adding the discrete topology; let $\underline{\mathbf{M}}' = \langle M; \mathrm{End}\,\underline{\mathbf{M}}\rangle$ be obtained from $\underset{\sim}{\mathbf{M}}$ by removing the discrete topology and let $\mathcal{A}' = \mathbb{ISP}\underline{\mathbf{M}}'$. The Two-For-One Strong Duality Theorem tells us that $\underset{\sim}{\mathbf{M}}'$ yields a strong duality on $\mathcal{A}'$, and therefore closed substructures of powers of $\underset{\sim}{\mathbf{M}}'$ are term-closed. Applying this fact to $\mathbf{Q} \leq \underset{\sim}{\mathbf{M}}'$, we see that $\mathbf{Q}$ is the intersection of the equalizers of a collection of unary term functions of $\underline{\mathbf{M}}'$, that is, of endomorphisms of $\underline{\mathbf{M}}$.

Now assume (b). From the Exclusively Unary Total Structure Theorem we see that $\underset{\sim}{\mathbf{M}}' = \langle M; G, R, \mathcal{T}\rangle$ yields a strong duality on $\mathcal{A}$ where G is the set of endomorphisms of $\underline{\mathbf{M}}$ and R is the set of subalgebras of $\underline{\mathbf{M}}$ viewed as unary relations. To see that $\underset{\sim}{\mathbf{M}} = \langle M; G, \mathcal{T}\rangle$ also yields a strong duality on $\mathcal{A}$ we must, by the $\underset{\sim}{\mathbf{M}}$-Shift Strong Duality Theorem, check that any endomorphism preserving operation on M also preserves subalgebras of $\underline{\mathbf{M}}$. But this is an immediate consequence of (iii). □

For completeness, we conclude with a well known observation which characterizes strong dualities that satisfy the ultimate constraint on the type of the dual structures.

PRIMAL ALGEBRA THEOREM 5.7. *The following are equivalent*:
(a) *the structure* $\underset{\sim}{\mathbf{M}}$ *with no operations, partial operations or relations yields a strong duality on* $\mathcal{A}$;
(b) $\underline{\mathbf{M}}$ *is a primal algebra.*

Proof. By the First Duality Theorem (Davey and Werner [13, 1.8], or see Clark and Davey [5, 1.5]), $\underline{\mathbf{M}}$ is primal in case $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathcal{A}$. Conversely, we use the NU-Strong-Duality Theorem to prove that (b) implies (a). □

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