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# THE COMPLEXITY OF DUALISABILITY: THREE-ELEMENT UNARY ALGEBRAS

D. M. CLARK
*Department of Mathematics, SUNY, New Paltz, NY 12561, USA*
*clarkd@newpaltz.edu*

B. A. DAVEY* and J. G. PITKETHLY†
*La Trobe University, Victoria 3086, Australia*
**B.Davey@latrobe.edu.au*
*†janegp@alphalink.com.au*

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We solve the dualisability problem in the class of three-element unary algebras. Our aim in tackling this class is to demonstrate the difficulty of the general dualisability problem. We also want to investigate the extent to which the dualisability of a finite algebra is a finiteness condition on the quasi-variety it generates.

*Keywords*: Natural duality; dualisability; ghost elements; unary algebras.

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## 0. Introduction

Perhaps the most fundamental problem in the theory of natural dualities is the **dualisability problem**: deciding exactly which finite algebras generate a quasi-variety that admits a natural duality. One of the goals of this paper is to demonstrate the difficulty of this problem. To do this, we solve the dualisability problem restricted to a class of apparently very simple algebras: three-element unary algebras. We find that the complexity of dualisability is evident even within this humble class.

Another goal of this paper is to give some insight into what the dualisability of an algebra can tell us about the structure of the quasi-variety that it generates. The dualisability of an algebra seems to be related to certain finiteness properties of the quasi-variety it generates. Some of our dualisability proofs make use of very strong finiteness conditions. However, we also find dualisable and non-dualisable algebras where the difference between the quasi-varieties they generate appears to be slight.

In the last chapter of the text by Clark and Davey [1], a complete solution is given to the dualisability problem for two-element algebras. The characterisation follows comparatively easily from the general results developed throughout the text. Unfortunately, the general tools of this text do not apply so readily to three-element unary algebras. Without many general tools to rely on, we will establish dualities from scratch, using our knowledge of the structure of the algebras in the quasi-varieties.

Unary algebras are actually rather complicated from the point of view of duality theory. Amongst the two-element algebras, it is the highly structured algebras which have simple dualising structures, and the simple algebras which have complicated dualising structures. For example, the two-element implicative lattice $\langle\{0,1\};\vee,\wedge,\rightarrow\rangle$ is dualised by the discrete pointed set $\langle\{0,1\};1,\mathcal{T}\rangle$, and the two-element pointed set $\langle\{0,1\};1\rangle$ is dualised by the discrete implicative lattice $\langle\{0,1\};\vee,\wedge,\rightarrow,\mathcal{T}\rangle$. The dualising structures that we shall obtain for three-element unary algebras will be quite unwieldy. But we are not trying to create useful dualities. Instead, we are aiming to shed some light on the general dualisability problem.

The dualisability of a three-element unary algebra is strongly related to the number of patterns that its unary term functions have. To make this more precise, consider a unary algebra $\underline{\mathbf{M}}$. A **kernel of** $\underline{\mathbf{M}}$ is an equivalence relation on $M$ of the form $\ker(u)$, for some unary term function $u$ of $\underline{\mathbf{M}}$ that is neither a constant map nor a permutation. We shall say that $\underline{\mathbf{M}}$ is an $n$**-kernel unary algebra** if $n$ is the number of different kernels of $\underline{\mathbf{M}}$.

There are only three non-trivial partitions of a three-element set. So the class of three-element unary algebras divides into zero-, one-, two- and three-kernel algebras. The main result of this paper is the following characterisation of dualisability for three-element unary algebras. We denote a unary operation $u:\{0,1,2\}\rightarrow\{0,1,2\}$ by the string $u(0)u(1)u(2)$.

**Dualisable three-element unary algebras**. *Let* $\underline{\mathbf{M}}$ *be a three-element unary algebra.*

(i) *If* $\underline{\mathbf{M}}$ *is a zero-kernel or one-kernel algebra, then* $\underline{\mathbf{M}}$ *is dualisable.*

(ii) *Assume that* $\underline{\mathbf{M}}$ *is a two-kernel algebra, on the set* $\{0,1,2\}$, *with kernels* $\{01|2\}$ *and* $\{02|1\}$. *Then* $\underline{\mathbf{M}}$ *is dualisable if and only if the none of the following conditions holds*:

  (a) *ppq and pqp are term functions of* $\underline{\mathbf{M}}$, *for some* $p,q\in\{0,1,2\}$ *with* $p\neq q$, *but* 010 *or* 002 *is not a term function of* $\underline{\mathbf{M}}$;

  (b) 010, 001 *and* 110 *are term functions of* $\underline{\mathbf{M}}$, *but* 222 *is not*;

  (c) 002, 020 *and* 202 *are term functions of* $\underline{\mathbf{M}}$, *but* 111 *is not.*

(iii) *If* $\underline{\mathbf{M}}$ *is a three-kernel algebra, then* $\underline{\mathbf{M}}$ *is not dualisable.*

Table 1. Six three-element unary algebras.

| Operations | Kernels | Dualisable? |
|---|---|---|
| $F_1 = \{012, 000, 001\}$ | 1 | Yes |
| $F_2 = \{012, 000, 001, 010\}$ | 2 | No |
| $F_3 = \{012, 000, 001, 010, 002\}$ | 2 | Yes |
| $F_4 = \{012, 000, 001, 010, 002, 110, 111\}$ | 2 | No |
| $F_5 = \{012, 000, 001, 010, 002, 110, 111, 222\}$ | 2 | Yes |
| $F_6 = \{012, 000, 001, 010, 002, 110, 111, 222, 011\}$ | 3 | No |

We will show that, given any two-kernel three-element unary algebra $\underline{\mathbf{M}}$, there is a straightforward method for constructing an isomorphic copy of $\underline{\mathbf{M}}$, on the set $\{0, 1, 2\}$, that has kernels $\{01|2\}$ and $\{02|1\}$. (See Lemma 4.1 and the discussion preceding it.) So the previous result really does completely characterise dualisability for three-element unary algebras.

Broadly speaking, our characterisation says that three-element unary algebras with few unary term functions are dualisable, while those with many unary term functions are non-dualisable. However, the dualisable and non-dualisable two-kernel algebras are tightly entangled. Indeed, there is a chain $F_1 \subseteq F_2 \subseteq \cdots \subseteq F_6$ of sets of unary operations on $\{0, 1, 2\}$ such that the corresponding algebras

$$\langle \{0,1,2\}; F_1 \rangle, \quad \langle \{0,1,2\}; F_2 \rangle, \ldots, \quad \langle \{0,1,2\}; F_6 \rangle$$

are alternatively dualisable and non-dualisable! The definitions of the operation sets $F_1, F_2, \ldots, F_6$ are given in Table 1. For each $i \in \{1, \ldots, 6\}$, the reader can check that every unary term function of the algebra $\langle \{0,1,2\}; F_i \rangle$ belongs to $F_i$, and so use our main theorem to determine whether or not this algebra is dualisable.

A three-element unary algebra with universe $\{0, 1, 2\}$ is determined, up to term equivalence, by its monoid of unary term functions. It turns out that there are exactly 699 such monoids on $\{0, 1, 2\}$. These 699 monoids determine 160 non-isomorphic unary algebras. In Tables 2 and 3, we indicate how the 699 monoids on $\{0, 1, 2\}$ and the 160 non-isomorphic unary algebras they determine are

Table 2. The 699 monoids on $\{0, 1, 2\}$.

| Kernels | 0 | 1 | 2 | 3 | Total |
|---|---|---|---|---|---|
| Dualisable | 24 | 198 | 210 | | 432 |
| Non-dualisable | | | 126 | 141 | 267 |

Table 3. The 160 non-isomorphic three-element unary algebras.

| Kernels | 0 | 1 | 2 | 3 | Total |
|---|---|---|---|---|---|
| Dualisable | 12 | 44 | 43 | | 99 |
| Non-dualisable | | | 24 | 37 | 61 |

distributed among the different kernel types. A computer-generated list was used to obtain the numbers.

We use only one previously established result to obtain dualities for three-element unary algebras. In [2], it is shown that a finite unary algebra $\underline{\mathbf{M}} = \langle M; F\rangle$ is dualisable if $F$ is a set of endomorphisms of some lattice $\mathbf{M}_0 = \langle M; \vee, \wedge\rangle$. These lattice-endomorphism algebras are scattered among the zero-, one- and two-kernel three-element unary algebras.

The research for this paper began with the authors pouring over a list of all 699 monoids on $\{0, 1, 2\}$, establishing dualisability or non-dualisability one algebra at a time. Order gradually grew out of this process. However, we find that a surprising number of very different arguments still seems to be required to solve the dualisability problem for three-element unary algebras.

After a preliminary section giving background and basic facts, we examine the four kernel types section by section. We shall use both general and particular results to complete our classification.

## 1. Preliminaries

In this section, we give a brief overview of the theory of natural dualities. All missing details and motivations may be found in the text by Clark and Davey [1]. We start by fixing our attention on a finite algebra $\underline{\mathbf{M}}$. An **algebraic relation on** $\underline{\mathbf{M}}$ is a relation $r \subseteq M^n$ which forms a subalgebra of $\underline{\mathbf{M}}^n$, for some $n \in \mathbb{N}$. An **algebraic operation on** $\underline{\mathbf{M}}$ is a homomorphism $g : \underline{\mathbf{M}}^n \to \underline{\mathbf{M}}$, for some $n \in \mathbb{N} \cup \{0\}$. An **algebraic partial operation on** $\underline{\mathbf{M}}$ is a homomorphism $h : \mathbf{D} \to \underline{\mathbf{M}}$ such that $\mathbf{D}$ is a subalgebra of $\underline{\mathbf{M}}^n$, for some $n \in \mathbb{N}$. We say that a topological structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ is an **alter ego of** $\underline{\mathbf{M}}$ if $G$ is a set of operations, $H$ is a set of partial operations and $R$ is a set of relations, each of which is algebraic on $\underline{\mathbf{M}}$, and $\mathcal{T}$ is the discrete topology on $M$. We want to try to use the alter ego $\underset{\sim}{\mathbf{M}}$ of $\underline{\mathbf{M}}$ to represent the algebras in the quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ as algebras of continuous homomorphisms.

Given $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, we define its **dual**, $\mathrm{D}(\mathbf{A})$, to be the set $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$, of all homomorphisms from $\mathbf{A}$ to $\underline{\mathbf{M}}$, regarded as a substructure of $\underset{\sim}{\mathbf{M}}^A$. Thus $\mathrm{D}(\mathbf{A})$ belongs to the category $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_c\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$ of all isomorphic copies of topologically closed substructures of non-zero powers of $\underset{\sim}{\mathbf{M}}$. Similarly, the **dual**, $\mathrm{E}(\mathbf{X})$, of a structure $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$ is the set $\boldsymbol{\mathfrak{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ regarded as a subalgebra of $\underline{\mathbf{M}}^X$. There is a natural evaluation map $e_{\mathbf{A}} : \mathbf{A} \to \mathrm{ED}(\mathbf{A})$, given by $e_{\mathbf{A}}(a)(x) := x(a)$, for all $a \in A$ and each $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. The map $e_{\mathbf{A}}$ is an embedding, since $\mathbf{A}$ is separated by homomorphisms into $\underline{\mathbf{M}}$. For each $a \in A$, we say that the map $e_{\mathbf{A}}(a)$ is **given by evaluation at** $a$ and that $e_{\mathbf{A}}(a)$ is an **evaluation**. For each $Y \subseteq \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ and $a \in A$, a map $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ is **given by evaluation at** $a$ **on** $Y$ if $\alpha{\restriction}_Y = e_{\mathbf{A}}(a){\restriction}_Y$. A subset $B$ of $A$ is a **support** for a map $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ if $\alpha(x) = \alpha(y)$, for all $x, y \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ such that $x{\restriction}_B = y{\restriction}_B$.

We say that $\underset{\sim}{\mathbf{M}}$ **yields a duality on A**, or that $G \cup H \cup R$ **yields a duality on A**, if $e_{\mathbf{A}}$ is surjective and is therefore an isomorphism from $\mathbf{A}$ onto the algebra of all continuous homomorphisms from its dual, $\mathrm{D}(\mathbf{A})$, into $\underset{\sim}{\mathbf{M}}$. Thus, $\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathbf{A}$ if every morphism $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ is an evaluation. The structure $\underset{\sim}{\mathbf{M}}$ **yields a duality on $\boldsymbol{\mathcal{A}}$** if $e_{\mathbf{A}}$ is surjective, for all $\mathbf{A} \in \boldsymbol{\mathcal{A}}$. If $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$, then we also say that $\underset{\sim}{\mathbf{M}}$ **dualises** $\underline{\mathbf{M}}$. Finally, if there is some alter ego $\underset{\sim}{\mathbf{M}}$ of $\underline{\mathbf{M}}$ which dualises $\underline{\mathbf{M}}$, then we say that $\underline{\mathbf{M}}$ is **dualisable**.

There are examples of both dualisable and non-dualisable unary algebras. Every finite unary algebra with only one fundamental operation is dualisable, by [10]. (See also [11].) In [2], it is shown that a finite unary algebra $\langle M; F\rangle$ is dualisable if $F$ is a set of endomorphisms of a lattice on the set $M$, or if $F$ is the set of all endomorphisms of a cyclic group on $M$. A family of non-dualisable unary algebras has been given by Heindorf [9]: a finite unary algebra $\langle M; F\rangle$, with $\{0, 1, 2\} \subseteq M$, is not dualisable if $F$ contains every map from $M$ into $\{0, 1\}$.

For each $n \in \mathbb{N}$, let $R_n$ denote the set of all $n$-ary algebraic relations on $\underline{\mathbf{M}}$. The following general result appears explicitly as [2, Lemma 1]. It can also be extracted from [1, Chap. 10].

**Lemma 1.1.** *Let $\underline{\mathbf{M}}$ be a finite algebra and let $n \in \mathbb{N}$. Let $\mathbf{A}$ be an algebra in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ and let $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$. Then $\alpha$ preserves (the relations in) $R_n$ if and only if $\alpha$ agrees with an evaluation on each subset of $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with at most $n$ elements.*

We will rely on an important theorem of Willard and Zadori, which gives a condition for a duality on the finite members of $\boldsymbol{\mathcal{A}}$ to lift to all of $\boldsymbol{\mathcal{A}}$. We say that the alter ego $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ of $\underline{\mathbf{M}}$ has **finite type** if $G \cup H \cup R$ is finite.

**Duality Compactness Theorem 1.2 (Willard [3], Zádori [13], [1]).** *Let $\underline{\mathbf{M}}$ be a finite algebra and let $\underset{\sim}{\mathbf{M}}$ be an alter ego of $\underline{\mathbf{M}}$ with finite type. Then $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ provided it yields a duality on each finite algebra in $\boldsymbol{\mathcal{A}}$.*

## 2. Dividing Unary Algebras into Petals

The zero-kernel unary algebras are exactly those whose fundamental operations are all constants or permutations. In this section, we prove that every finite zero-kernel

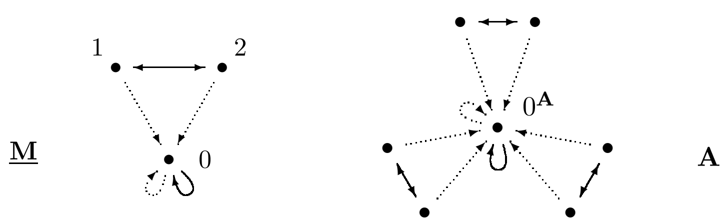


Fig. 1. Examples 2.1 and 2.2.

unary algebra (of any size) is dualisable. To do this, we show that finite zero-kernel algebras generate extremely simple quasi-varieties. Each of these quasi-varieties can be created, via coproducts, from a finite number of building-block algebras. The following example gives an easy illustration of this phenomenon.

**Example 2.1.** Consider the three-element unary algebra $\underline{\mathbf{M}} := \langle \{0,1,2\}; 021, 000\rangle$, shown in Fig. 1. The quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ is determined by the quasi-equations

$$021 \circ 021(x) \approx x\,, \quad 000(x) \approx 000(y) \quad \text{and} \quad 021(x) \approx x \iff x \approx 000(x)\,.$$

Figure 1 gives an example of a typical algebra $\mathbf{A}$ from $\boldsymbol{\mathcal{A}}$. It is easy to check that every algebra in $\boldsymbol{\mathcal{A}}$ is a coproduct of copies of $\underline{\mathbf{M}}$.

Let $\underline{\mathbf{M}} = \langle M; F\rangle$ be an arbitrary unary algebra and let $\mathbf{A}$ belong to $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$. There is a directed graph naturally associated with the algebra $\mathbf{A}$. We define the graph $G(\mathbf{A}) = \langle A; E_{\mathbf{A}}\rangle$ such that

$$E_{\mathbf{A}} := \{(a, u(a)) \mid a \in A \text{ and } u \in F\}\,.$$

Now define the **centre of A** to be the subuniverse of $\mathbf{A}$ given by

$$C_{\mathbf{A}} := \{m^{\mathbf{A}} \mid m \text{ is the value of a constant term function of } \underline{\mathbf{M}}\}\,.$$

Let $G^*(\mathbf{A})$ denote the induced subgraph of $G(\mathbf{A})$ with vertex set $A\backslash C_{\mathbf{A}}$. (The $^*$ is meant to serve as a reminder that $G^*(\mathbf{A})$ is obtained by removing part of $G(\mathbf{A})$.) A subalgebra $\mathbf{P}$ of $\mathbf{A}$ is called a **petal of A** if $P\backslash C_{\mathbf{A}}$ is the vertex set of a connected component of the graph $G^*(\mathbf{A})$. An algebra $\mathbf{P}$ is a **petal of** $\boldsymbol{\mathcal{A}}$ if $\mathbf{P}$ is a petal of some $\mathbf{A} \in \boldsymbol{\mathcal{A}}$.

Note that the centre and petals of a particular unary algebra depend on which quasi-variety we choose as its home. Whenever we are talking about centres and petals, we shall make sure that the chosen quasi-variety is clear.

The following example demonstrates the origin of the names "petal" and "centre".

**Example 2.2.** Again, consider the algebra $\underline{\mathbf{M}} := \langle \{0,1,2\}; 021, 000\rangle$. The only element of $\underline{\mathbf{M}}$ that is the value of a constant term function is 0. In Fig. 1, the algebra $\mathbf{A}$, from $\mathbb{ISP}(\underline{\mathbf{M}})$, has been drawn to resemble a flower. This algebra has centre $C_{\mathbf{A}} = \{0^{\mathbf{A}}\}$ and three petals.

**Lemma 2.3.** *Let* $\underline{\mathbf{M}}$ *be a unary algebra and let* $\mathbf{A}$ *be a non-trivial algebra in* $\mathbb{ISP}(\underline{\mathbf{M}})$. *Then* $\mathbf{A}$ *is the coproduct* (*in* $\mathbb{ISP}(\underline{\mathbf{M}})$) *of its petals.*

**Proof.** Define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$. First assume that $A = C_{\mathbf{A}}$. Then $\mathbf{A}$ has no petals. Choose an algebra $\mathbf{B}$ from $\boldsymbol{\mathcal{A}}$. There is a unique homomorphism $x : \mathbf{A} \to \mathbf{B}$, given by $x(m^{\mathbf{A}}) = m^{\mathbf{B}}$, for each $m \in M$ that is the value of a constant term function of $\underline{\mathbf{M}}$. (To see that $x$ is well defined, let $m_1, m_2 \in M$, with $m_1 \neq m_2$, such that both $m_1$ and $m_2$ are values of constant term functions of $\underline{\mathbf{M}}$. Since the non-trivial algebra $\mathbf{A}$ is separated by homomorphisms into $\underline{\mathbf{M}}$, we must have $m_1^{\mathbf{A}} \neq m_2^{\mathbf{A}}$.) Thus $\mathbf{A}$ is the empty-indexed coproduct in $\boldsymbol{\mathcal{A}}$.

Now assume that $A \neq C_{\mathbf{A}}$ and let $\{\mathbf{P}_i \mid i \in I\}$ be the set of petals of $\mathbf{A}$. Then $C_{\mathbf{A}} \subseteq P_i$, for all $i \in I$, and $\{P_i \backslash C_{\mathbf{A}} \mid i \in I\}$ is a partition of $A \backslash C_{\mathbf{A}}$. Let $\mathbf{B} \in \boldsymbol{\mathcal{A}}$ and, for each $i \in I$, let $x_i : \mathbf{P}_i \to \mathbf{B}$ be a homomorphism. We must have $x_i(m^{\mathbf{A}}) = m^{\mathbf{B}}$, for all $i \in I$ and $m^{\mathbf{A}} \in C_{\mathbf{A}}$. So the homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$, given by $x := \bigcup\{x_i \mid i \in I\}$, is the unique extension of the family $\{x_i \mid i \in I\}$. It follows that $\mathbf{A}$ is the coproduct of $\{\mathbf{P}_i \mid i \in I\}$ in $\boldsymbol{\mathcal{A}}$. □

Within classes of non-unary algebras, most algebras cannot be written as a coproduct of simpler algebras. Similarly, in many quasi-varieties of unary algebras, the petals are just as complicated as the algebras in general. Nevertheless, there are quasi-varieties of unary algebras whose petals are particularly well behaved. If the petals of $\boldsymbol{\mathcal{A}} = \mathbb{ISP}(\underline{\mathbf{M}})$ are simple, it makes sense to establish a duality for $\boldsymbol{\mathcal{A}}$ in two steps. First, find a structure that yields a duality on the petals of $\boldsymbol{\mathcal{A}}$, and then enrich this structure so that it yields a duality on all of $\boldsymbol{\mathcal{A}}$. The following two lemmas show how the second step can be done.

**Lemma 2.4.** *Let $\underline{\mathbf{M}}$ be a finite unary algebra and define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$. Let $\mathbf{A}$ be a finite algebra in $\boldsymbol{\mathcal{A}}$, with $A \neq C_{\mathbf{A}}$, and assume that $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ preserves $R_4$. Then there is a petal $\mathbf{P}$ of $\mathbf{A}$ such that $P$ is a support for $\alpha$.*

**Proof.** Let $\mathbf{P}_1, \dots, \mathbf{P}_n$ be the petals of $\mathbf{A}$, where $n \in \mathbb{N}$. (Since $A \neq C_{\mathbf{A}}$, the algebra $\mathbf{A}$ has at least one petal.) If the map $\alpha$ is constant, then every subset of $A$ is a support for $\alpha$. So we can assume that there exist $w, x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with $\alpha(w) \neq \alpha(x)$. We will define a sequence $w_0, \dots, w_n$ of homomorphisms in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ such that $w_i$ and $w_{i+1}$ agree on $A \backslash P_{i+1}$, for every $i \in \{0, \dots, n-1\}$. First define $w_0 := w$. For each $i \in \{0, \dots, n-1\}$, we can define the homomorphism $w_{i+1} : \mathbf{A} \to \underline{\mathbf{M}}$ by $w_{i+1} := w_i{\restriction}_{A \backslash P_{i+1}} \cup x{\restriction}_{P_{i+1}}$, since $\mathbf{P}_{i+1}$ is a cofactor of $\mathbf{A}$. As $A = P_1 \cup \dots \cup P_n$, we have $w_n = x$.

Since $\alpha(w_0) = \alpha(w) \neq \alpha(x) = \alpha(w_n)$, there is some $j \in \{0, \dots, n-1\}$ with $\alpha(w_j) \neq \alpha(w_{j+1})$. To see that $P_{j+1}$ is a support for $\alpha$, let $y, z \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with $y{\restriction}_{P_{j+1}} = z{\restriction}_{P_{j+1}}$. The map $\alpha$ preserves $R_4$. So, by Lemma 1.1, there is some $a \in A$ such that $\alpha$ is given by evaluation at $a$ on $\{w_j, w_{j+1}, y, z\}$. The maps $w_j$ and $w_{j+1}$ can only differ on $P_{j+1}$. As $w_j(a) = \alpha(w_j) \neq \alpha(w_{j+1}) = w_{j+1}(a)$, it follows that $a \in P_{j+1}$. Thus $\alpha(y) = y(a) = z(a) = \alpha(z)$, whence $P_{j+1}$ is a support for $\alpha$. □

**Lemma 2.5.** *Let $\underline{\mathbf{M}}$ be a finite unary algebra and let $n \in \mathbb{N}$ with $n \geqslant 2$. Assume that $R_n$ yields a duality on every finite petal of $\mathbb{ISP}(\underline{\mathbf{M}})$. Then $\underset{\sim}{\mathbf{M}} := \langle M; R_{n+2}, \mathcal{T} \rangle$ yields a duality on $\mathbb{ISP}(\underline{\mathbf{M}})$.*

**Proof.** We will be using the Duality Compactness Theorem 1.2. Let $\mathbf{A}$ be a finite algebra in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ and let $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ be a morphism. We want to prove that $\alpha$ is an evaluation. So we can assume that $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ is not empty. Choose some $z \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. For every petal $\mathbf{P}$ of $\mathbf{A}$ and each $w \in \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$, we can define the homomorphism $\bar{w} \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ by $\bar{w} := w \cup z{\restriction}_{A \backslash P}$, since $\mathbf{P}$ is a cofactor of $\mathbf{A}$. Define the map $\alpha_{\mathbf{P}} : \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}}) \to M$ by $\alpha_{\mathbf{P}}(w) := \alpha(\bar{w})$, for each petal $\mathbf{P}$ of $\mathbf{A}$.

First assume that $A = C_{\mathbf{A}}$. Then there is a unique homomorphism from $\mathbf{A}$ to $\underline{\mathbf{M}}$, and so $|\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})| = 1$. Since $\alpha$ preserves $R_{n+2}$, it follows by Lemma 1.1 that $\alpha$ is an evaluation.

Now assume that $A \neq C_{\mathbf{A}}$ and $\alpha$ is not constant. There are $x_1, x_2 \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with $\alpha(x_1) \neq \alpha(x_2)$. By Lemma 2.4, there is a petal $\mathbf{P}$ of $\mathbf{A}$ such that $P$ is a support for $\alpha$. We have $y_1 := x_1{\restriction}_P \cup z{\restriction}_{A \backslash P}$ and $y_2 := x_2{\restriction}_P \cup z{\restriction}_{A \backslash P}$ in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with $\alpha(y_1) = \alpha(x_1) \neq \alpha(x_2) = \alpha(y_2)$.

We want to show that $\alpha_{\mathbf{P}} : \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}}) \to M$ preserves $R_n$. By Lemma 1.1, it is enough to show that $\alpha_{\mathbf{P}}$ agrees with an evaluation on every subset of $\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$ with at most $n$ elements. Let $w_1, \dots, w_n \in \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$. Since $\alpha$ preserves $R_{n+2}$, the map $\alpha$ is given by evaluation at some $a \in A$ on $\{y_1, y_2, \overline{w_1}, \dots, \overline{w_n}\}$, by Lemma 1.1. We must have $a \in P$, since $y_1(a) = \alpha(y_1) \neq \alpha(y_2) = y_2(a)$ and $y_1{\restriction}_{A \backslash P} = y_2{\restriction}_{A \backslash P}$. Now, for all $i \in \{1, \dots, n\}$, we have $\alpha_{\mathbf{P}}(w_i) = \alpha(\overline{w_i}) = \overline{w_i}(a) = w_i(a)$. So $\alpha_{\mathbf{P}}$ is given by evaluation at $a$ on $\{w_1, \dots, w_n\}$, whence $\alpha_{\mathbf{P}}$ preserves $R_n$.

As $R_n$ yields a duality on the finite petal $\mathbf{P}$, there exists $b \in P$ such that $\alpha_{\mathbf{P}}$ is given by evaluation at $b$. Since $P$ is a support for $\alpha$, we have

$$\alpha(x) = \alpha(\overline{x{\restriction}_P}) = \alpha_{\mathbf{P}}(x{\restriction}_P) = x{\restriction}_P(b) = x(b)\,,$$

for all $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. So $\alpha$ is an evaluation.

Finally, assume that $A \neq C_{\mathbf{A}}$ and that $\alpha$ is constant. Suppose that, for every petal $\mathbf{P}$ of $\mathbf{A}$, the map $\alpha_{\mathbf{P}}$ is not an evaluation. Now let $\mathbf{P}$ be any petal of $\mathbf{A}$. As $R_n$ yields a duality on $\mathbf{P}$ and $\alpha_{\mathbf{P}}$ is not an evaluation, the map $\alpha_{\mathbf{P}}$ does not preserve $R_n$. By Lemma 1.1, there must be homomorphisms $w_{\mathbf{P}1}, \dots, w_{\mathbf{P}n} \in \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$ such that $\alpha_{\mathbf{P}}$ does not agree with an evaluation on $\{w_{\mathbf{P}1}, \dots, w_{\mathbf{P}n}\}$. By Lemma 2.3, the algebra $\mathbf{A}$ is the coproduct of its petals. This implies that, for each $i \in \{1, \dots, n\}$, we can define the homomorphism

$$w_i := \bigcup \{ w_{\mathbf{P}i} \mid \mathbf{P} \text{ is a petal of } \mathbf{A} \}$$

in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. Since $\alpha$ preserves $R_{n+2}$, the map $\alpha$ is given by evaluation at some $a \in A$ on $\{w_1, \dots, w_n\}$. Let $\mathbf{Q}$ be a petal of $\mathbf{A}$ containing $a$. We are assuming that $\alpha$ is constant. So, for each $i \in \{1, \dots, n\}$, we have $\alpha_{\mathbf{Q}}(w_{\mathbf{Q}i}) = \alpha(\overline{w_{\mathbf{Q}i}}) = \alpha(w_i) = w_i(a) = w_{\mathbf{Q}i}(a)$. Thus $\alpha_{\mathbf{Q}}$ is given by evaluation at $a$ on $\{w_{\mathbf{Q}1}, \dots, w_{\mathbf{Q}n}\}$, which is a contradiction.

We have shown that there is a petal $\mathbf{P}$ of $\mathbf{A}$ for which $\alpha_{\mathbf{P}}$ is an evaluation. Let $b \in P$ such that $\alpha_{\mathbf{P}}$ is given by evaluation at $b$. Then, since $\alpha$ is constant, we have $\alpha(x) = \alpha_{\mathbf{P}}(x{\restriction}_P) = x{\restriction}_P(b) = x(b)$, for every $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. Thus $\alpha$ is an evaluation. Hence $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, by the Duality Compactness Theorem. ☐

The previous lemma can be used to show very easily that every finite zero-kernel unary algebra is dualisable. In order to apply the lemma, we first show that the quasi-variety generated by a finite zero-kernel algebra is extremely simple.

**Lemma 2.6.** *Let $\underline{\mathbf{M}}$ be a finite unary algebra and define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$.*

(i) *If $\underline{\mathbf{M}}$ is a zero-kernel algebra, then, up to isomorphism, the quasi-variety $\boldsymbol{\mathcal{A}}$ has finitely many petals, all of which are finite.*

(ii) *If $\underline{\mathbf{M}}$ is not a zero-kernel algebra, then $\boldsymbol{\mathcal{A}}$ has both arbitrarily large finite petals and infinite petals.*

**Proof.** Assume that $\underline{\mathbf{M}}$ is a zero-kernel algebra. Let $F_{\mathrm{p}}$ be the set of all unary term functions of $\underline{\mathbf{M}}$ that are permutations, and let $F_{\mathrm{c}}$ be the set of all constant unary term functions of $\underline{\mathbf{M}}$. Then every unary term function of $\underline{\mathbf{M}}$ belongs to $F_{\mathrm{p}} \cup F_{\mathrm{c}}$. For each $u \in F_{\mathrm{p}}$, the map $u^{-1} : M \to M$ is a term function of $\underline{\mathbf{M}}$, as $\underline{\mathbf{M}}$ is finite. Now let $\mathbf{P}$ be a petal of $\boldsymbol{\mathcal{A}}$. Since the graph $G^*(\mathbf{P})$ is connected, it follows that $P \backslash C_{\mathbf{P}}$ is an orbit of $F_{\mathrm{p}}$. Therefore $|P| = |C_{\mathbf{P}}| + |P \backslash C_{\mathbf{P}}| \leqslant |F_{\mathrm{c}}| + |F_{\mathrm{p}}|$. The quasi-variety $\boldsymbol{\mathcal{A}}$ is locally finite. So, up to isomorphism, there are only finitely many petals of $\boldsymbol{\mathcal{A}}$, all of which are finite.

Now assume that $\underline{\mathbf{M}}$ is not a zero-kernel algebra. There is a unary term function $u$ of $\underline{\mathbf{M}}$ that is neither a constant map nor a permutation. So there are distinct elements $a$, $b$ and $c$ of $M$ such that $u(a) = u(b) \neq u(c)$. Let $S$ be a non-empty set and define the algebra $\mathbf{A} := \underline{\mathbf{M}}^S \times \underline{\mathbf{M}}$ in $\boldsymbol{\mathcal{A}}$. Now define the subset $X := \{a, b\}^S \times \{c\}$ of $A$. Then $u(x) = u(y) \notin C_{\mathbf{A}}$, for all $x, y \in X$. It follows that $X$ is a subset of the vertices of a connected component of the graph $G^*(\mathbf{A})$. So $X$ is contained in a petal of $\mathbf{A}$. Thus $\boldsymbol{\mathcal{A}}$ has both arbitrarily large finite petals and infinite petals. □

**Theorem 2.7.** *Every finite zero-kernel unary algebra is dualisable.*

**Proof.** Let $\underline{\mathbf{M}}$ be a finite zero-kernel unary algebra and define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$. Then Lemma 2.6 tells us that, up to isomorphism, there are only finitely many petals of $\boldsymbol{\mathcal{A}}$, all of which are finite. So there is some $n \in \mathbb{N}\backslash\{1\}$ such that $|\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})| \leqslant n$, for each petal $\mathbf{P}$ of $\boldsymbol{\mathcal{A}}$. By Lemma 1.1, it follows that $R_n$ yields a duality on every petal of $\boldsymbol{\mathcal{A}}$. Thus $\underline{\mathbf{M}}$ is dualisable, by Lemma 2.5. □

The following result, though not necessary for our characterisation of dualisable three-element unary algebras, is another simple corollary of Lemma 2.5. We will present further applications of Lemma 2.5 in the next section. For each unary algebra $\underline{\mathbf{M}} = \langle M; F \rangle$, with $\star \notin M$, we can define the **one-point extension of $\underline{\mathbf{M}}$** to be the extension $\underline{\mathbf{M}}_\star := \langle M \cup \{\star\}; F \rangle$ of $\underline{\mathbf{M}}$, where $u(\star) = \star$, for all $u \in F$.

**Theorem 2.8.** *A finite unary algebra is dualised by an alter ego with finite type if and only if its one-point extension is dualised by an alter ego with finite type.*

**Proof.** Let $\underline{\mathbf{M}}$ be a finite unary algebra, with $\star \notin M$, and let $\underline{\mathbf{M}}_\star$ be the one-point extension of $\underline{\mathbf{M}}$. Define the two quasi-varieties $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ and $\boldsymbol{\mathcal{A}}_\star := \mathbb{ISP}(\underline{\mathbf{M}}_\star)$. For each $n \in \mathbb{N}$, let $R_n$ denote the set of all $n$-ary algebraic relations on $\underline{\mathbf{M}}$, and let $R_n^\star$ denote the set of all $n$-ary algebraic relations on $\underline{\mathbf{M}}_\star$.

First assume that $\underline{\mathbf{M}}$ is dualised by an alter ego $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ with finite type. There is some $n \in \mathbb{N}$ such that the operations in $G \cup H$ have arity at most $n-1$ and the relations in $R$ have arity at most $n$. It is easy to check that $R_n$ yields a duality on each algebra in $\boldsymbol{\mathcal{A}}$ (see [1, Sec. 2.1.2]). We will prove that $R_n^\star$ yields a duality on every finite petal of $\boldsymbol{\mathcal{A}}_\star$. By Lemma 2.5, it will then follow that $\underline{\mathbf{M}}_\star$ is dualised by an alter ego with finite type.

Let $\mathbf{P}$ be a finite petal of $\boldsymbol{\mathcal{A}}_\star$ and assume that $\alpha : \boldsymbol{\mathcal{A}}_\star(\mathbf{P}, \underline{\mathbf{M}}_\star) \to M \cup \{\star\}$ preserves $R_n^\star$. We want to show that $\alpha$ is an evaluation. The algebra $\mathbf{P}$ is separated by homomorphisms into $\underline{\mathbf{M}}_\star$. For each $x \in \boldsymbol{\mathcal{A}}_\star(\mathbf{P}, \underline{\mathbf{M}}_\star)$, we have $x(P) \subseteq M$ or $x(P) = \{\star\}$, since the graph $G(\mathbf{P})$ is connected. So $\mathbf{P}$ is separated by homomorphisms into $\underline{\mathbf{M}}$, and therefore $\mathbf{P} \in \boldsymbol{\mathcal{A}}$. Note that every homomorphism in $\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$ also belongs to $\boldsymbol{\mathcal{A}}_\star(\mathbf{P}, \underline{\mathbf{M}}_\star)$. The set $M$ is a unary algebraic relation on $\underline{\mathbf{M}}_\star$. So $\alpha$ preserves $M$, and therefore $\alpha(x) \in M$, for all $x \in \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$. We can now define $\beta : \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}}) \to M$ by $\beta := \alpha{\restriction}_{\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})}$. As $\alpha$ preserves $R_n^\star$ and $R_n \subseteq R_n^\star$, the map $\beta$ preserves $R_n$. Since $R_n$ yields a duality on $\mathbf{P} \in \boldsymbol{\mathcal{A}}$, there is some $a \in P$ such that $\beta$ is given by evaluation at $a$. Choose any $x \in \boldsymbol{\mathcal{A}}_\star(\mathbf{P}, \underline{\mathbf{M}}_\star)$. If $x(P) \subseteq M$, then $\alpha(x) = \beta(x) = x(a)$. If $x(P) = \{\star\}$, then $\alpha(x) = \star = x(a)$, as $\alpha$ preserves the unary relation $\{\star\}$. Thus $\alpha$ is an evaluation, whence $\underline{\mathbf{M}}_\star$ is dualised by an alter ego with finite type.

Now assume that $\underline{\mathbf{M}}_\star$ is dualised by an alter ego with finite type. There is some $n \in \mathbb{N}$ such that $R_n^\star$ yields a duality on each algebra in $\boldsymbol{\mathcal{A}}_\star$. We want to show that $R_n$ yields a duality on every finite petal of $\boldsymbol{\mathcal{A}}$. Let $\mathbf{P}$ be a finite petal of $\boldsymbol{\mathcal{A}}$ and let $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}}) \to M$ preserve $R_n$. Since $\underline{\mathbf{M}} \in \boldsymbol{\mathcal{A}}_\star$, we know that $\mathbf{P} \in \boldsymbol{\mathcal{A}}_\star$. As the graph $G(\mathbf{P})$ is connected, we must have $x(P) \subseteq M$ or $x(P) = \{\star\}$, for all $x \in \boldsymbol{\mathcal{A}}_\star(\mathbf{P}, \underline{\mathbf{M}}_\star)$. Define $\beta : \boldsymbol{\mathcal{A}}_\star(\mathbf{P}, \underline{\mathbf{M}}_\star) \to M \cup \{\star\}$ by

$$\beta(x) = \begin{cases} \alpha(x) & \text{if } x(P) \subseteq M\,, \\ \star & \text{if } x(P) = \{\star\}\,. \end{cases}$$

By Lemma 1.1, the map $\alpha$ agrees with an evaluation on every subset of $\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$ with at most $n$ elements. It is now easy to see that $\beta$ agrees with an evaluation on every subset of $\boldsymbol{\mathcal{A}}_\star(\mathbf{P}, \underline{\mathbf{M}}_\star)$ with at most $n$ elements. So, using Lemma 1.1 again, the map $\beta$ preserves $R_n^\star$. Since $R_n^\star$ yields a duality on $\boldsymbol{\mathcal{A}}_\star$, this implies that $\beta$ is an evaluation. Thus $\alpha = \beta{\restriction}_{\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})}$ is an evaluation. Using Lemma 2.5, it follows that $\underline{\mathbf{M}}$ is dualised by an alter ego with finite type. ◻

Davey and Knox [6] have given a condition under which the one-point extension of a dualisable non-unary algebra is dualisable. The previous theorem complements this result. At the moment, it is not known whether every dualisable algebra can be dualised by an alter ego of finite type. This is called the Finite Type Problem (see [1, Sec. 2.2.13]).

## 3. One-Kernel Unary Algebras

One-kernel algebras are almost as simple as zero-kernel algebras. In this section, we show that the quasi-variety generated by a finite one-kernel unary algebra $\underline{\mathbf{M}}$

satisfies a very strong finiteness condition. There is a finite set $\mathcal{B}$ of petals of $\mathbb{ISP}(\underline{\mathbf{M}})$ such that every finite petal of $\mathbb{ISP}(\underline{\mathbf{M}})$ is nearly isomorphic to a petal from $\mathcal{B}$, in a sense that we shall later make explicit. We will then use this finiteness condition to help prove that every finite one-kernel algebra is dualisable.

We begin by proving an easy, useful lemma.

**Lemma 3.1.** *Let $\underline{\mathbf{M}}$ be a one-kernel unary algebra with kernel $\theta$. Then every unary term function of $\underline{\mathbf{M}}$ preserves $\theta$.*

**Proof.** There must be a unary term function $u$ of $\underline{\mathbf{M}}$ with $\ker(u) = \theta$. The map $u$ is neither constant nor a permutation. Now let $v$ be any unary term function of $\underline{\mathbf{M}}$. If $v$ is constant or $\ker(v) = \theta$, then $v$ preserves $\theta$. So we can assume that $v$ is a permutation. Let $(a, b) \in \theta$. The term function $u \circ v$ of $\underline{\mathbf{M}}$ is neither constant nor a permutation. So $\ker(u \circ v) = \theta$. This gives us $u \circ v(a) = u \circ v(b)$, and therefore $(v(a), v(b)) \in \theta$. Thus $v$ preserves $\theta$. □

Now consider a finite unary algebra $\mathbf{A}$. For each $a \in A$, define $\mathrm{sg}_{\mathbf{A}}(a)$ to be the subuniverse of $\mathbf{A}$ generated by $a$. We say that $a \in A$ is an **outer element of A** if $\mathrm{sg}_{\mathbf{A}}(a)$ is a maximal one-generated subuniverse of $\mathbf{A}$. The members of $A$ that are not outer elements of $\mathbf{A}$ are called **inner elements of A**. Define $A_{\mathrm{out}}$ to be the set of all outer elements of $\mathbf{A}$, and $A_{\mathrm{in}}$ to be the set of all inner elements of $\mathbf{A}$. Then $A_{\mathrm{out}}$ is a generating set for $\mathbf{A}$, and $A_{\mathrm{in}}$ is a subuniverse of $\mathbf{A}$.

A surjection $\varphi : \mathbf{A} \twoheadrightarrow \mathbf{B}$ is said to be **gentle** if $\varphi$ is one-to-one on $A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)$, for all $a \in A_{\mathrm{out}}$. Let $\mathcal{B}$ be a set of finite petals of $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$. Then $\mathcal{B}$ is called a **gentle basis for $\boldsymbol{\mathcal{A}}$** if, for each finite petal $\mathbf{P}$ of $\boldsymbol{\mathcal{A}}$, there is a gentle surjection $\varphi : \mathbf{P} \twoheadrightarrow \mathbf{B}$, for some $\mathbf{B} \in \mathcal{B}$. These definitions are illustrated by the following example.

**Example 3.2.** Consider the three-element unary algebra $\underline{\mathbf{M}} := \langle \{0, 1, 2\}; 001 \rangle$. The quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ is determined by the equation $001^2(x) \approx 001^2(y)$. The algebra $\mathbf{A}$, given in Fig. 2, is a typical member of $\boldsymbol{\mathcal{A}}$. One of the petals of $\mathbf{A}$ is marked with a dotted line. It is easy to see that each finite petal of $\boldsymbol{\mathcal{A}}$ has a gentle surjection onto $\mathbf{B}_1$ or $\mathbf{B}_2$, from Fig. 2. So $\{\mathbf{B}_1, \mathbf{B}_2\}$ is a gentle basis for $\boldsymbol{\mathcal{A}}$.

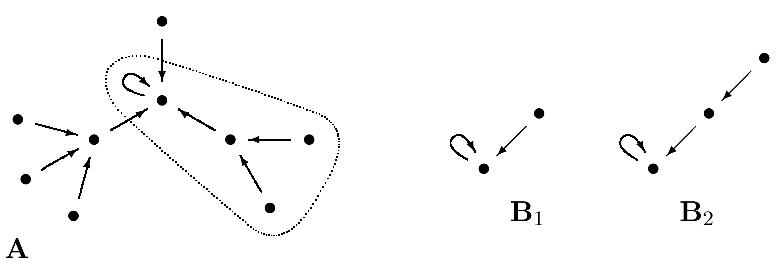


Fig. 2. Example 3.2.

Gentle surjections are so called because they do not destroy too much of the structure of the algebra they act on. Loosely speaking, gentle surjections can only collapse repeated structure on the outside of the algebra. This is the case, for example, for the gentle surjection from the marked petal of $\mathbf{A}$ onto $\mathbf{B}_2$.

**Lemma 3.3.** *Let* $\mathbf{A}$ *be a finite unary algebra and let* $\varphi : \mathbf{A} \twoheadrightarrow \mathbf{B}$ *be a gentle surjection. Then* $\varphi$ *is a retraction and, for each subalgebra* $\mathbf{C}$ *of* $\mathbf{A}$ *such that* $\varphi{\restriction}_C$ *is one-to-one, there is a coretraction* $\psi : \mathbf{B} \hookrightarrow \mathbf{A}$ *for* $\varphi$ *with* $C \subseteq \psi(B)$*.*

**Proof.** Let $\mathbf{C}$ be a subalgebra of $\mathbf{A}$ such that $\varphi{\restriction}_C$ is one-to-one. (Such subalgebras of $\mathbf{A}$ do exist. For example, we could choose $\mathbf{sg}_{\mathbf{A}}(a)$, for any $a \in A$.) Define the equivalence relation $\equiv_\varphi$ on $A_{\mathrm{out}}$ by

$$a \equiv_\varphi b \iff \varphi(\mathrm{sg}_{\mathbf{A}}(a)) = \varphi(\mathrm{sg}_{\mathbf{A}}(b))\,.$$

We want to find a transversal $\mathcal{T}$ of the blocks of $\equiv_\varphi$ for which $C \cap A_{\mathrm{out}} \subseteq \mathrm{sg}_{\mathbf{A}}(\mathcal{T})$. To see that this is possible, let $a, b \in C \cap A_{\mathrm{out}}$ and assume that $a \equiv_\varphi b$. Then $\varphi(a) \in \varphi(\mathrm{sg}_{\mathbf{A}}(a)) = \varphi(\mathrm{sg}_{\mathbf{A}}(b))$ and $\varphi(b) \in \varphi(\mathrm{sg}_{\mathbf{A}}(b)) = \varphi(\mathrm{sg}_{\mathbf{A}}(a))$. Since $\varphi{\restriction}_C$ is one-to-one, this implies that $a \in \mathrm{sg}_{\mathbf{A}}(b)$ and $b \in \mathrm{sg}_{\mathbf{A}}(a)$, whence $\mathrm{sg}_{\mathbf{A}}(a) = \mathrm{sg}_{\mathbf{A}}(b)$. So there is a transversal $\mathcal{T}$ of $A_{\mathrm{out}}/{\equiv_\varphi}$ such that $C \cap A_{\mathrm{out}} \subseteq \mathrm{sg}_{\mathbf{A}}(\mathcal{T})$. This gives us $C \subseteq A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(\mathcal{T})$.

Since $\varphi$ is gentle, the homomorphism $\varphi{\restriction}_{A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)}$ is one-to-one, for each $a \in \mathcal{T}$. We wish to show that $\psi : \mathbf{B} \to \mathbf{A}$, given by

$$\psi := \bigcup\{(\varphi{\restriction}_{A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)})^{-1} \mid a \in \mathcal{T}\}\,,$$

is a well-defined homomorphism. We must have

$$B = \varphi(A) = \varphi(A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(\mathcal{T})) = \bigcup\{\varphi(A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)) \mid a \in \mathcal{T}\}\,.$$

Now let $a, b \in \mathcal{T}$ with $a \neq b$. We need to check that the maps $(\varphi{\restriction}_{A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)})^{-1}$ and $(\varphi{\restriction}_{A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(b)})^{-1}$ agree on $\varphi(A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)) \cap \varphi(A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(b))$. So let $c_a \in A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)$ and $c_b \in A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(b)$ such that $\varphi(c_a) = \varphi(c_b)$. We shall prove that $c_a = c_b$. Then we will have

$$(\varphi{\restriction}_{A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)})^{-1}(\varphi(c_a)) = c_a = c_b = (\varphi{\restriction}_{A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(b)})^{-1}(\varphi(c_b))\,.$$

First, suppose that $c_a$ is an outer element of $\mathbf{A}$. Then $c_b$ is also an outer element of $\mathbf{A}$, as $\varphi$ is one-to-one on $A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)$. It follows that $\mathrm{sg}_{\mathbf{A}}(c_a) = \mathrm{sg}_{\mathbf{A}}(a)$ and $\mathrm{sg}_{\mathbf{A}}(c_b) = \mathrm{sg}_{\mathbf{A}}(b)$. But this gives us

$$\varphi(\mathrm{sg}_{\mathbf{A}}(a)) = \varphi(\mathrm{sg}_{\mathbf{A}}(c_a)) = \mathrm{sg}_{\mathbf{B}}(\varphi(c_a)) = \mathrm{sg}_{\mathbf{B}}(\varphi(c_b)) = \varphi(\mathrm{sg}_{\mathbf{A}}(c_b)) = \varphi(\mathrm{sg}_{\mathbf{A}}(b))\,,$$

and therefore $a \equiv_\varphi b$, which is a contradiction. So $c_a$ is an inner element of $\mathbf{A}$. Since $\varphi$ is one-to-one on $A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(b)$, this implies that $c_a = c_b$. It follows that $\psi$ is a well-defined homomorphism.

To prove that $\psi$ is a coretraction for $\varphi$, let $b \in B$. There is some $a \in \mathcal{T}$ with $b \in \varphi(A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a))$. We have $\varphi \circ \psi(b) = \varphi \circ (\varphi{\restriction}_{A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(a)})^{-1}(b) = b$. So $\psi$ is a coretraction for $\varphi$, with $C \subseteq A_{\mathrm{in}} \cup \mathrm{sg}_{\mathbf{A}}(\mathcal{T}) = \psi(B)$. □

We can now clarify the statements we made in the introduction to this section. Consider a gentle surjection $\varphi : \mathbf{A} \twoheadrightarrow \mathbf{B}$. For each $a \in A$, the map $\varphi$ is one-to-one on $\mathrm{sg}_{\mathbf{A}}(a)$. So, for every $a \in A$, there is a coretraction $\psi_a : \mathbf{B} \hookrightarrow \mathbf{A}$ for $\varphi$ with $a \in \psi_a(B)$, by Lemma 3.3. Therefore $\mathbf{A}$ can be covered with the images of coretractions for $\varphi$. This tells us that $\mathbf{B}$ retains most of the structure of $\mathbf{A}$. So gentle surjections really are "nearly isomorphisms", and having a finite gentle basis is a very strong finiteness condition on a quasi-variety. We will prove that the quasi-variety generated by a finite one-kernel algebra must have a finite gentle basis.

**Example 3.4.** Define the unary algebra $\underline{\mathbf{M}} := \langle \{0,1,2\}; F\rangle$, where

$$F := \{012, 021\} \cup \{pqq \mid p, q \in M\}\,.$$

Then $\underline{\mathbf{M}}$ is a one-kernel algebra, with kernel $\{0|12\}$. For each $n \in \mathbb{N}$, every petal of $\underline{\mathbf{M}}^n$ has associated with it a partition of the set $\{1,\ldots,n\}$ with at most two blocks. We will illustrate this using $n = 5$. The centre of the algebra $\underline{\mathbf{M}}^5$ is the set of all constant strings $\{\widehat{0}, \widehat{1}, \widehat{2}\}$. Consider the element $(1,2,1,0,0)$ of $M^5$. The petal of $\underline{\mathbf{M}}^5$ containing $(1,2,1,0,0)$ has the underlying set $P := P_\bullet \cup P_\circ$, where

$$P_\bullet := \{(a,a,a,b,b) \mid a, b \in M\} \quad \text{and} \quad P_\circ := (\{1,2\}^3 \times \{0\}^2) \cup (\{0\}^3 \times \{1,2\}^2)\,.$$

To see this, note that the operations in $F\backslash\{012, 021\}$ collectively map each member of $P_\circ$ onto $P_\bullet$. The petal $\mathbf{P}$ has associated with it the partition $\{123|45\}$ of $\{1,2,3,4,5\}$. It is easy to check that $P_\circ = P_{\mathrm{out}}$ and $P_\bullet = P_{\mathrm{in}}$. So the petal $\mathbf{P}$ of $\underline{\mathbf{M}}^5$ has $3^2 = 9$ inner elements. Indeed, every finite petal of $\mathbb{ISP}(\underline{\mathbf{M}})$ has at most 9 inner elements.

**Lemma 3.5.** *Let $\underline{\mathbf{M}}$ be a finite one-kernel unary algebra and define $n := |M|$. Then each finite petal of $\mathbb{ISP}(\underline{\mathbf{M}})$ has at most $n! + n$ inner elements.*

**Proof.** Let $\theta$ be the kernel of $\underline{\mathbf{M}}$ and let $\mathbf{P}$ be a finite petal of $\mathbb{ISP}(\underline{\mathbf{M}})$. Every subalgebra of $\mathbf{P}$ contains the centre $C_{\mathbf{P}}$. So we can assume that $C_{\mathbf{P}} \subseteq P_{\mathrm{in}}$, since otherwise we have $|P| = |C_{\mathbf{P}}| \leqslant |M| = n$.

We can also assume that $\mathbf{P} \leqslant \underline{\mathbf{M}}^S$, for some non-empty set $S$. Every element $a \in P$ determines two partitions of $S$:

$$\mathcal{P}(a) := \{a^{-1}(m) \mid m \in M\}\backslash\{\emptyset\} \quad \text{and} \quad \mathcal{P}_\theta(a) := \{a^{-1}(m/\theta) \mid m \in M\}\backslash\{\emptyset\}\,.$$

We begin by proving that there is a partition $\mathcal{Q}$ of $S$ such that $\mathcal{P}(a) = \mathcal{Q}$, for all $a \in P_{\mathrm{in}}\backslash C_{\mathbf{P}}$.

Choose some $b \in P_{\mathrm{out}}$. Define the partition $\mathcal{Q} := \mathcal{P}_\theta(b)$ of $S$ and the subset

$$P_{\mathcal{Q}} := \{a \in P_{\mathrm{in}}\backslash C_{\mathbf{P}} \mid \mathcal{P}(a) = \mathcal{Q}\} \cup \{a \in P_{\mathrm{out}} \mid \mathcal{P}_\theta(a) = \mathcal{Q}\}$$

of $P\backslash C_{\mathbf{P}}$. We wish to show that $P_{\mathcal{Q}} = P\backslash C_{\mathbf{P}}$. Since $\mathbf{P}$ is a petal, the graph $G^*(\mathbf{P})$ is connected. So it is enough to show that $P_{\mathcal{Q}}$ determines a connected component of $G^*(\mathbf{P})$. Let $a \in P\backslash C_{\mathbf{P}}$ and let $u$ be a unary term function of $\underline{\mathbf{M}}$ such that $u(a) \notin C_{\mathbf{P}}$.

We will show that $a \in P_{\mathcal{Q}}$ if and only if $u(a) \in P_{\mathcal{Q}}$. As $b \in P_{\mathcal{Q}}$, it will then follow that $P_{\mathcal{Q}} = P \backslash C_{\mathbf{P}}$.

**Case 1.** $u(a) \in P_{\text{out}}$. Since $\text{sg}_{\mathbf{P}}(u(a)) \subseteq \text{sg}_{\mathbf{P}}(a)$ and $u(a)$ is an outer element of $\mathbf{P}$, we must have $\text{sg}_{\mathbf{P}}(u(a)) = \text{sg}_{\mathbf{P}}(a)$ and $a \in P_{\text{out}}$. There is a unary term function $v$ of $\underline{\mathbf{M}}$ such that $v \circ u(a) = a$. The maps $u$ and $v$ both preserve $\theta$, by Lemma 3.1. So $\mathcal{P}_\theta(a) = \mathcal{P}_\theta(u(a))$. Thus $a \in P_{\mathcal{Q}}$ if and only if $u(a) \in P_{\mathcal{Q}}$.

**Case 2.** $a \in P_{\text{out}}$ and $u(a) \in P_{\text{in}}$. We must have $a \notin \text{sg}_{\mathbf{P}}(u(a))$. Therefore the term function $u$ of $\underline{\mathbf{M}}$ is not a permutation. (Otherwise, the finiteness of $\underline{\mathbf{M}}$ implies that $u^{-1}$ is a term function of $\underline{\mathbf{M}}$.) Since $u(a) \notin C_{\mathbf{P}}$, the term function $u$ of $\underline{\mathbf{M}}$ cannot be constant. Therefore $\ker(u) = \theta$, and it follows that $\mathcal{P}(u(a)) = \mathcal{P}_\theta(a)$. So $a \in P_{\mathcal{Q}}$ if and only if $u(a) \in P_{\mathcal{Q}}$.

**Case 3.** $a \in P_{\text{in}}$. This implies that $u(a) \in P_{\text{in}}$. The algebra $\mathbf{P}$ is generated by its outer elements. So there is some $c \in P_{\text{out}}$ and a unary term function $v$ of $\underline{\mathbf{M}}$ such that $v(c) = a$. By the previous case, we have $c \in P_{\mathcal{Q}}$ if and only if $v(c) \in P_{\mathcal{Q}}$. The previous case also tells us that $c \in P_{\mathcal{Q}}$ if and only if $u \circ v(c) \in P_{\mathcal{Q}}$. Therefore $a \in P_{\mathcal{Q}}$ if and only if $u(a) \in P_{\mathcal{Q}}$.

We have shown that $\mathcal{P}(a) = \mathcal{Q}$, for all $a \in P_{\text{in}} \backslash C_{\mathbf{P}}$. So an element of $P_{\text{in}} \backslash C_{\mathbf{P}}$ must take a different value out of $M$ on each block of the partition $\mathcal{Q}$. Since $\mathcal{Q} := \mathcal{P}_\theta(b)$, the number of different blocks of $\mathcal{Q}$ is at most $|M/\theta| \leqslant |M| = n$. Therefore $|P_{\text{in}} \backslash C_{\mathbf{P}}| \leqslant n!$, which gives us $|P_{\text{in}}| = |C_{\mathbf{P}}| + |P_{\text{in}} \backslash C_{\mathbf{P}}| \leqslant n + n!$. □

**Lemma 3.6.** *Let* $\underline{\mathbf{M}}$ *be a finite one-kernel unary algebra. Then* $\mathbb{ISP}(\underline{\mathbf{M}})$ *has a finite gentle basis.*

**Proof.** Define $n := |M|$. Then the one-generated free algebra in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ has at most $n^n$ elements. So there is some $k \in \mathbb{N}$ such that, up to isomorphism, there are only $k$ one-generated algebras in $\boldsymbol{\mathcal{A}}$. Now let $\mathbf{P}$ be a finite petal of $\boldsymbol{\mathcal{A}}$. We will show that there is a gentle surjection from $\mathbf{P}$ onto a petal $\mathbf{B}$ of $\boldsymbol{\mathcal{A}}$ with $|B| \leqslant n^n k(n! + n)!$. It will then follow that $\boldsymbol{\mathcal{A}}$ has a finite gentle basis.

Let $\mathcal{S}$ be the set of all maximal one-generated subuniverses of $\mathbf{P}$. Then $\mathcal{S} = \{\text{sg}_{\mathbf{P}}(a) \mid a \in P_{\text{out}}\}$. Define the equivalence relation $\equiv$ on $\mathcal{S}$ such that $S \equiv T$ if and only if there is an isomorphism from $\mathbf{S}$ to $\mathbf{T}$ that fixes each element of $S \cap P_{\text{in}}$. Now choose a transversal $\mathcal{T}$ of the blocks of $\equiv$ and define the subuniverse $B := \cup\mathcal{T}$ of $\mathbf{P}$.

For each $(S, T) \in \equiv$, choose an isomorphism $\eta_{ST} : \mathbf{S} \hookrightarrow\!\!\!\!\to \mathbf{T}$ that fixes each element of $S \cap P_{\text{in}}$. We want to define a gentle surjection $\varphi : \mathbf{P} \twoheadrightarrow \mathbf{B}$ by

$$\varphi := \bigcup \{\eta_{ST} \mid T \in \mathcal{T} \text{ and } S \in T/\equiv\}.$$

First, note that

$$P = \cup\mathcal{S} = \bigcup \{S \mid T \in \mathcal{T} \text{ and } S \in T/\equiv\}.$$

Now consider $\eta_{S_1T_1}$ and $\eta_{S_2T_2}$ such that $S_1 \neq S_2$, where $T_i \in \mathcal{T}$ and $S_i \in T_i/\equiv$, for each $i \in \{1,2\}$. Since $S_1$ is a maximal one-generated subuniverse of $\mathbf{P}$, the subuniverse $S_1 \cap S_2$ of $\mathbf{P}$ is properly contained in $S_1$. So $S_1 \cap S_2 \subseteq P_{\text{in}}$, and therefore $\eta_{S_1T_1}$ and $\eta_{S_2T_2}$ agree on $S_1 \cap S_2$. It follows that $\varphi$ is a well-defined surjection, with $\varphi{\restriction}_{P_{\text{in}}} = \text{id}_{P_{\text{in}}}$.

To check that $\varphi$ is gentle, let $a \in P_{\text{out}}$. Then $S := \text{sg}_{\mathbf{P}}(a)$ belongs to $\mathcal{S}$, and $S \in T/\equiv$, for some $T \in \mathcal{T}$. We want to show that $\varphi$ is one-to-one on $P_{\text{in}} \cup \text{sg}_{\mathbf{P}}(a)$. We have $\varphi{\restriction}_{P_{\text{in}} \cup \text{sg}_{\mathbf{P}}(a)} = \varphi{\restriction}_{P_{\text{in}} \cup S} = \text{id}_{P_{\text{in}}} \cup \eta_{ST}$. Since $\eta_{ST}$ is one-to-one, it is enough to show that $\eta_{ST}(S \backslash P_{\text{in}}) \subseteq T \backslash P_{\text{in}}$. The maximality of $S$ and $T$ guarantees that $S_{\text{out}} = S \cap P_{\text{out}}$ and $T_{\text{out}} = T \cap P_{\text{out}}$. As $\eta_{ST}$ is an isomorphism, we get $\eta_{ST}(S \backslash P_{\text{in}}) = \eta_{ST}(S_{\text{out}}) = T_{\text{out}} = T \backslash P_{\text{in}}$. Thus $\varphi$ is one-to-one on $P_{\text{in}} \cup \text{sg}_{\mathbf{P}}(a)$, whence $\varphi$ is a gentle surjection.

We now want to prove that $\mathbf{B}$ is a petal of $\boldsymbol{\mathcal{A}}$. To do this, we begin by showing that $\varphi^{-1}(C_{\mathbf{B}}) = C_{\mathbf{P}}$. Clearly, $C_{\mathbf{P}} \subseteq \varphi^{-1}(C_{\mathbf{B}})$. So let $a \in \varphi^{-1}(C_{\mathbf{B}})$. By Lemma 3.3, there is a coretraction $\psi : \mathbf{B} \hookrightarrow \mathbf{P}$ for $\varphi$ such that $a \in \psi(B)$. This implies that $a = \psi \circ \varphi(a) \in \psi(C_{\mathbf{B}}) \subseteq C_{\mathbf{P}}$. Thus $\varphi^{-1}(C_{\mathbf{B}}) = C_{\mathbf{P}}$. Since $\mathbf{P}$ is a petal of $\boldsymbol{\mathcal{A}}$, the graph $G^*(\mathbf{P})$ is connected. As $\varphi : \mathbf{P} \twoheadrightarrow \mathbf{B}$ is a surjective homomorphism with $\varphi^{-1}(C_{\mathbf{B}}) = C_{\mathbf{P}}$, it follows that the graph $G^*(\mathbf{B})$ is also connected. So $\mathbf{B}$ is a petal of $\boldsymbol{\mathcal{A}}$.

It remains to show that $|B| \leqslant n^n k(n! + n)!$. Each one-generated algebra in $\boldsymbol{\mathcal{A}}$ has at most $n^n$ elements. Therefore $|B| = |\cup \mathcal{T}| \leqslant n^n |\mathcal{T}|$. Each $T \in \mathcal{T}$ determines a one-generated algebra $\mathbf{T}$ from $\boldsymbol{\mathcal{A}}$. We know that, up to isomorphism, there are only $k$ one-generated algebras in $\boldsymbol{\mathcal{A}}$. So we want to bound the number of members of $\mathcal{T}$ that can determine isomorphic algebras. Let $\mathbf{A}$ be any one-generated algebra in $\boldsymbol{\mathcal{A}}$ and assume that $T_1, T_2 \in \mathcal{T}$, with $T_1 \neq T_2$, such that there are isomorphisms $\varphi_1 : \mathbf{A} \hookrightarrow\!\!\!\!\rightarrow \mathbf{T}_1$ and $\varphi_2 : \mathbf{A} \hookrightarrow\!\!\!\!\rightarrow \mathbf{T}_2$. Then $\varphi_2 \circ \varphi_1^{-1} : \mathbf{T}_1 \hookrightarrow\!\!\!\!\rightarrow \mathbf{T}_2$ is an isomorphism. Since $T_1 \not\equiv T_2$, we must have $\varphi_2 \circ \varphi_1^{-1}(a) \neq a$, for some $a \in T_1 \cap P_{\text{in}}$. As $\varphi_1(A_{\text{in}}) = (T_1)_{\text{in}} = T_1 \cap P_{\text{in}}$, this gives us $b := \varphi_1^{-1}(a) \in A_{\text{in}}$ with $\varphi_1(b) \neq \varphi_2(b)$. So the embeddings $\varphi_1{\restriction}_{A_{\text{in}}} : \mathbf{A}_{\text{in}} \hookrightarrow \mathbf{P}_{\text{in}}$ and $\varphi_2{\restriction}_{A_{\text{in}}} : \mathbf{A}_{\text{in}} \hookrightarrow \mathbf{P}_{\text{in}}$ are different. Using the previous lemma, we know that $|P_{\text{in}}| \leqslant n!+n$. So the number of different embeddings from $\mathbf{A}_{\text{in}}$ into $\mathbf{P}_{\text{in}}$ is at most $|P_{\text{in}}|! \leqslant (n! + n)!$. This implies that there are at most $(n! + n)!$ subuniverses $T$ of $\mathbf{P}$ belonging to $\mathcal{T}$ such that $\mathbf{T}$ is isomorphic to $\mathbf{A}$. Finally, as there are exactly $k$ non-isomorphic one-generated algebras in $\boldsymbol{\mathcal{A}}$, we get $|B| \leqslant n^n |\mathcal{T}| \leqslant n^n k(n! + n)!$. □

The following lemma shows that a finite unary algebra must be a zero-kernel or one-kernel algebra in order to generate a quasi-variety with a finite gentle basis.

**Lemma 3.7.** *Let* $\underline{\mathbf{M}}$ *be a finite unary algebra with at least two kernels. Then* $\mathbb{ISP}(\underline{\mathbf{M}})$ *does not have a finite gentle basis.*

**Proof.** There are unary term functions $u$ and $v$ of $\underline{\mathbf{M}}$, neither of which is a constant map or a permutation, such that $\ker(u) \nsubseteq \ker(v)$. So there are elements $a, b \in M$

with $u(a) = u(b)$ and $v(a) \neq v(b)$. As $u$ is not constant and $v$ is not a permutation, there exist $c, d, e \in M$, with $d \neq e$, such that $u(b) \neq u(c)$ and $v(d) = v(e)$.

Now let $n \in \mathbb{N}$. We will construct a finite petal $\mathbf{P}$ of $\mathbb{ISP}(\underline{\mathbf{M}})$ with $|P_{\text{in}}| \geqslant 2^n$. Each gentle surjection from $\mathbf{P}$ must be one-to-one on $P_{\text{in}}$. So it will then follow that $\mathbb{ISP}(\underline{\mathbf{M}})$ does not have a finite gentle basis. Define the subset $X$ of $M^{n+3}$ by

$$X := \{a, b\}^n \times \{c\} \times \{d\} \times \{e\}.$$

For all $x, y \in X$, we have $u(x) = u(y) \notin C_{\underline{\mathbf{M}}^{n+3}}$, as $u(a) = u(b) \neq u(c)$. So $X$ is a subset of the vertices of a connected component of the graph $G^*(\underline{\mathbf{M}}^{n+3})$. Consequently, the set $X$ is contained in a petal $\mathbf{P}$ of $\underline{\mathbf{M}}^{n+3}$. For every $x \in X$, we have $x \notin \mathrm{sg}_{\mathbf{P}}(v(x))$, as $v(d) = v(e)$. Therefore $v(x) \in P_{\text{in}}$, for each $x \in X$. Since $v(a) \neq v(b)$, this implies that

$$|P_{\text{in}}| \geqslant |\{v(x) \mid x \in X\}| = 2^n.$$

Thus $\mathbb{ISP}(\underline{\mathbf{M}})$ does not have a finite gentle basis. □

Lemmas 2.6, 3.6 and 3.7 tell us exactly which finite unary algebras generate a quasi-variety with a finite gentle basis.

**Corollary 3.8.** *Let $\underline{\mathbf{M}}$ be a finite unary algebra. Then $\mathbb{ISP}(\underline{\mathbf{M}})$ has a finite gentle basis if and only if $\underline{\mathbf{M}}$ is a zero-kernel or one-kernel algebra.*

We can now finish this section by proving that each finite one-kernel unary algebra is dualisable.

**Theorem 3.9.** *Every finite one-kernel unary algebra is dualisable.*

**Proof.** Let $\underline{\mathbf{M}}$ be a finite one-kernel unary algebra. By Lemma 3.6, there is a finite gentle basis $\mathcal{B}$ for $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$. Since $\mathcal{B}$ consists of a finite number of finite algebras, we can choose some $n \in \mathbb{N}$ such that $n \geqslant |\boldsymbol{\mathcal{A}}(\mathbf{B}, \underline{\mathbf{M}})| + 3$, for all $\mathbf{B} \in \mathcal{B}$. We shall prove that $R_n$ yields a duality on every finite petal of $\boldsymbol{\mathcal{A}}$. By Lemma 2.5, it will then follow that $\underset{\sim}{\mathbf{M}} := \langle M; R_{n+2}, \mathcal{T}\rangle$ dualises $\underline{\mathbf{M}}$.

Let $\mathbf{P}$ be a finite petal of $\boldsymbol{\mathcal{A}}$. There is a gentle surjection $\varphi_0 : \mathbf{P} \twoheadrightarrow \mathbf{B}_0$, for some $\mathbf{B}_0 \in \mathcal{B}$. Since $\varphi_0{\restriction}_{P_{\text{in}}}$ is one-to-one, we can use Lemma 3.3 to find a coretraction $\psi_0 : \mathbf{B}_0 \hookrightarrow \mathbf{P}$ for $\varphi_0$ with $P_{\text{in}} \subseteq \psi_0(B_0)$. The subalgebra $\mathbf{B} := \psi_0(\mathbf{B}_0)$ of $\mathbf{P}$ is isomorphic to $\mathbf{B}_0$. There is a gentle surjection $\varphi : \mathbf{P} \twoheadrightarrow \mathbf{B}$, given by $\varphi := \psi_0 \circ \varphi_0$, such that $\varphi{\restriction}_{P_{\text{in}}} = \mathrm{id}_{P_{\text{in}}}$. Now let $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}}) \to M$ preserve $R_n$. We want to show that $\alpha$ is an evaluation.

First, assume that $\alpha(x) = \alpha(x \circ \varphi)$, for all $x \in \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$. Since $\mathbf{B}$ is isomorphic to an algebra in $\mathcal{B}$, we have $|\boldsymbol{\mathcal{A}}(\mathbf{B}, \underline{\mathbf{M}})| \leqslant n - 3$. As the map $\alpha$ preserves $R_n$, it follows by Lemma 1.1 that there is some $a \in P$ such that $\alpha$ is given by evaluation at $a$ on the subset $\{w \circ \varphi \mid w \in \boldsymbol{\mathcal{A}}(\mathbf{B}, \underline{\mathbf{M}})\}$ of $\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$. For all $x \in \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$, we have $\alpha(x) = \alpha(x \circ \varphi) = x \circ \varphi(a)$. So $\alpha$ is given by evaluation at $\varphi(a)$.

Now assume that there is some $y \in \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$ such that $\alpha(y) \neq \alpha(y \circ \varphi)$. Since $\mathbf{P}$ is finite, there is some $k \in \mathbb{N}$ with $P_{\mathrm{out}} = \{a_1, \dots, a_k\}$. We will define a sequence $y_0, \dots, y_k$ of homomorphisms in $\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$ such that $P_{\mathrm{in}} \subseteq \mathrm{eq}(y_i, y \circ \varphi)$ and $P \backslash \mathrm{sg}_{\mathbf{P}}(a_{i+1}) \subseteq \mathrm{eq}(y_i, y_{i+1})$, for all $i \in \{0, \dots, k-1\}$. As $\varphi{\restriction}_{P_{\mathrm{in}}} = \mathrm{id}_{P_{\mathrm{in}}}$, we can define $y_0 := y$. Now let $i \in \{0, \dots, k-1\}$ and assume that $y_i$ has been defined. For all $a \in P \backslash \mathrm{sg}_{\mathbf{P}}(a_{i+1})$, we have $\mathrm{sg}_{\mathbf{P}}(a) \cap \mathrm{sg}_{\mathbf{P}}(a_{i+1}) \subseteq P_{\mathrm{in}}$. Since $y_i$ and $y \circ \varphi$ agree on $P_{\mathrm{in}}$, this means that we can define the homomorphism

$$y_{i+1} := y_i{\restriction}_{P \backslash \mathrm{sg}_{\mathbf{P}}(a_{i+1})} \cup y \circ \varphi{\restriction}_{\mathrm{sg}_{\mathbf{P}}(a_{i+1})}$$

in $\boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$. As $P_{\mathrm{out}}$ is a generating set for $\mathbf{P}$, we get $y_n = y \circ \varphi$.

As $\alpha(y_0) = \alpha(y) \neq \alpha(y \circ \varphi) = \alpha(y_n)$, there is some $j \in \{0, \dots, k-1\}$ such that $\alpha(y_j) \neq \alpha(y_{j+1})$. By Lemma 1.1, there is some $b \in P$ for which $\alpha$ is given by evaluation at $b$ on $\{y_j, y_{j+1}\} \cup \{w \circ \varphi \mid w \in \boldsymbol{\mathcal{A}}(\mathbf{B}, \underline{\mathbf{M}})\}$. The maps $y_j$ and $y_{j+1}$ agree on $P \backslash \mathrm{sg}_{\mathbf{P}}(a_{j+1})$. Since $y_j(b) = \alpha(y_j) \neq \alpha(y_{j+1}) = y_{j+1}(b)$, we must have $b \in \mathrm{sg}_{\mathbf{P}}(a_{j+1})$. To see that $\alpha$ is given evaluation at $b$, let $x \in \boldsymbol{\mathcal{A}}(\mathbf{P}, \underline{\mathbf{M}})$. There is some $c \in \mathrm{sg}_{\mathbf{P}}(a_{j+1})$ such that $\alpha$ is given by evaluation at $c$ on the set $\{x, y_j, y_{j+1}\} \cup \{w \circ \varphi \mid w \in \boldsymbol{\mathcal{A}}(\mathbf{B}, \underline{\mathbf{M}})\}$. Since $\varphi$ is gentle, the map $\varphi$ is one-to-one on $\mathrm{sg}_{\mathbf{P}}(a_{j+1})$. So, by Lemma 3.3, there is a coretraction $\psi : \mathbf{B} \hookrightarrow \mathbf{P}$ for $\varphi$ such that $\mathrm{sg}_{\mathbf{P}}(a_{j+1}) \subseteq \psi(B)$. We have $\psi \circ \varphi{\restriction}_{\psi(B)} = \mathrm{id}_{\psi(B)}$ and $b, c \in \mathrm{sg}_{\mathbf{P}}(a_{j+1}) \subseteq \psi(B)$. Therefore

$$\alpha(x) = x(c) = x \circ \psi \circ \varphi(c) = \alpha(x \circ \psi \circ \varphi) = x \circ \psi \circ \varphi(b) = x(b)\,,$$

whence $\alpha$ is an evaluation. Hence $R_n$ yields a duality on every finite petal of $\boldsymbol{\mathcal{A}}$. □

## 4. Dualisable Two-Kernel Three-Element Unary Algebras

Were it not for the two-kernel algebras, the characterisation of dualisability for three-element unary algebras would be very simple. All of the zero- and one-kernel algebras are dualisable, and none of the three-kernel algebras is dualisable. It is only amongst the two-kernel algebras that the dualisable and non-dualisable algebras are hard to differentiate. We shall split the class of two-kernel three-element unary algebras into four further types. Two of these types will be exclusively dualisable, and the other two will be exclusively non-dualisable. First, we prove the claim, made in the introduction, that it suffices to consider the two-kernel algebras with kernels $\{01|2\}$ and $\{02|1\}$.

Isomorphic copies of a unary algebra can be created via **conjugation**. Consider a three-element unary algebra $\underline{\mathbf{M}} = \langle \{0,1,2\}; F \rangle$ and let $v : \{0,1,2\} \to \{0,1,2\}$ be a permutation. For each $u \in F$, define the unary operation ${}^{v}u$ on $\{0,1,2\}$ by ${}^{v}u := v \circ u \circ v^{-1}$. Then $v : M \to M$ is an isomorphism from $\underline{\mathbf{M}}$ onto the algebra ${}^{v}\underline{\mathbf{M}} := \langle \{0,1,2\}; {}^{v}F \rangle$, where ${}^{v}F := \{{}^{v}u \mid u \in F\}$. Furthermore, every isomorphic copy of $\underline{\mathbf{M}}$, on the set $\{0,1,2\}$, can be obtained via conjugation in this way.

**Lemma 4.1.** *Let* $\underline{\mathbf{M}}$ *be a two-kernel three-element unary algebra. There is an isomorphic copy of* $\underline{\mathbf{M}}$, *on the set* $\{0,1,2\}$, *that has kernels* $\{01|2\}$ *and* $\{02|1\}$.

**Proof.** Let $\underline{\mathbf{M}} = \langle\{0,1,2\}; F\rangle$ be a two-kernel unary algebra with kernels $\theta_1$ and $\theta_2$. There is some $m \in M$ for which $|m/\theta_1| = 2 = |m/\theta_2|$. We can assume that $m \neq 0$ since, otherwise, the algebra $\underline{\mathbf{M}}$ already has kernels $\{01|2\}$ and $\{02|1\}$. Let $v : M \to M$ be the transposition interchanging 0 and $m$. It is easy to check that the isomorphic copy ${}^v\underline{\mathbf{M}} = \langle\{0,1,2\}; {}^vF\rangle$ of $\underline{\mathbf{M}}$ has kernels $\{01|2\}$ and $\{02|1\}$. □

We will often be using symmetry to reduce our work load. Given any unary algebra $\underline{\mathbf{M}}$, on the set $\{0,1,2\}$, with kernels $\{01|2\}$ and $\{02|1\}$, the algebra ${}^{021}\underline{\mathbf{M}}$ is isomorphic to $\underline{\mathbf{M}}$ and also has kernels $\{01|2\}$ and $\{02|1\}$.

**Lemma 4.2.** *Let* $\underline{\mathbf{M}}$ *be a two-kernel unary algebra, on the set* $\{0,1,2\}$, *with kernels* $\{01|2\}$ *and* $\{02|1\}$. *Then the unary term functions of* $\underline{\mathbf{M}}$ *all belong to the set* $\{012, 021\} \cup \{ppq, pqp \mid p, q \in M\}$.

**Proof.** We just need to show that 012 and 021 are the only permutations that can be term functions of $\underline{\mathbf{M}}$. There exist unary term functions $u_1$ and $u_2$ of $\underline{\mathbf{M}}$ such that $\ker(u_1) = \{02|1\}$ and $\ker(u_2) = \{01|2\}$. Now let $v$ be a unary term function of $\underline{\mathbf{M}}$ that is a permutation. Then $u_1 \circ v$ is neither a constant map nor a permutation. So $\ker(u_1 \circ v) = \{01|2\}$ or $\ker(u_1 \circ v) = \{02|1\}$. In either case, we have $u_1 \circ v(1) \neq u_1 \circ v(2)$, which implies that $v(1) = 1$ or $v(2) = 1$. Symmetrically, we have $u_2 \circ v(1) \neq u_2 \circ v(2)$, which tells us that $v(1) = 2$ or $v(2) = 2$. It follows that $v = 012$ or $v = 021$. □

The following theorem introduces the four types of unary algebras with kernels $\{01|2\}$ and $\{02|1\}$. Each of these types is preserved under conjugation by 021. The two unary operations $f_1 := 010$ and $f_2 := 002$ on $\{0,1,2\}$ play a very important role in our characterisation.

**Theorem 4.3.** *Let* $\underline{\mathbf{M}}$ *be a two-kernel unary algebra, on the set* $\{0,1,2\}$, *with kernels* $\{01|2\}$ *and* $\{02|1\}$. *Let* $F$ *denote the set of all unary term functions of* $\underline{\mathbf{M}}$. *Then at least one of the following is true*:

$(2)_{\mathrm{O}}$ *each map in* $F$ *preserves the order* $\preccurlyeq$ *with* $1 \preccurlyeq 0 \preccurlyeq 2$;
$(2)_{\mathrm{P}}$ $\{f_1, f_2\} \nsubseteq F$, *and* $\{ppq, pqp\} \subseteq F$, *for some* $p, q \in M$ *with* $p \neq q$;
$(2)_{\mathrm{M}}$ $\{010, 001, 110\} \subseteq F$ *and* $222 \notin F$, *or* $\{002, 020, 202\} \subseteq F$ *and* $111 \notin F$;
$(2)_{\mathrm{R}}$ $\{f_1, f_2\} \subseteq F$, *and condition* $(2)_{\mathrm{M}}$ *fails.*

**Proof.** Assume that $\underline{\mathbf{M}}$ has neither type $(2)_{\mathrm{O}}$, type $(2)_{\mathrm{P}}$ nor type $(2)_{\mathrm{M}}$. To prove that $\underline{\mathbf{M}}$ has type $(2)_{\mathrm{R}}$, it suffices to show that $\{f_1, f_2\} \subseteq F$. Since $\underline{\mathbf{M}}$ does not have type $(2)_{\mathrm{P}}$, we can assume that $\{ppq, pqp\} \nsubseteq F$, for all $p, q \in M$ with $p \neq q$.

As $\underline{\mathbf{M}}$ does not have type $(2)_{\mathrm{O}}$, there is a map in $F$ that does not preserve the order $\preccurlyeq$. Using Lemma 4.2, the only maps that can belong to $F$ and that do not preserve $\preccurlyeq$ are 021, 221, 220, 001, 101, 121 and 020. Since $\{01|2\}$ and $\{02|1\}$ are kernels of $\underline{\mathbf{M}}$, there exist $p, q, r, s \in M$, with $p \neq q$ and $r \neq s$, such that $ppq \in F$ and

$rsr \in F$. Since $\{ppq, pqp\} \nsubseteq F$ and $\{rrs, rsr\} \nsubseteq F$, we have $pqp \notin F$ and $rrs \notin F$. This implies that 021, 121, 020 $\notin F$, since

$$ppq \circ 021 = ppq \circ 121 = ppq \circ 020 = pqp \notin F\,.$$

Similarly, we have $221, 001 \notin F$, as $rsr \circ 221 = rsr \circ 001 = rrs \notin F$. As $F$ contains a map that does not preserve $\preccurlyeq$, it follows that $101 \in F$ or $220 \in F$.

First, assume that $101 \in F$. Then $f_1 = 010 = 101 \circ 101 \in F$. So $110 \notin F$ and $001 \notin F$. We have $112 \notin F$ and $221 \notin F$, since $101 \circ 112 = 010 \circ 221 = 001 \notin F$. We know there is a map in $F$ with kernel $\{01|2\}$. It follows that $002 \in F$ or $220 \in F$. As $f_2 = 002 = 220 \circ 220$, we have $\{f_1, f_2\} \subseteq F$.

Now assume $220 \in F$. This case is symmetric, under conjugation by 021, to the case $101 \in F$. We have $101 = {}^{021}220 \in {}^{021}F$. By the previous case, it follows that $\{f_1, f_2\} \subseteq {}^{021}F$. So $\{f_1, f_2\} = \{{}^{021}f_2, {}^{021}f_1\} \subseteq F$. □

The names of the types in the previous theorem are meant to serve as *aide-mémoires*. Type-$(2)_{\mathrm{O}}$ algebras have Order-preserving operations; type-$(2)_{\mathrm{P}}$ algebras have operations with the Patterns *ppq* and *pqp*, for some $p, q \in M$; type-$(2)_{\mathrm{M}}$ algebras are Missing a constant operation; and the Rest are type-$(2)_{\mathrm{R}}$ algebras.

The dualisability of type-$(2)_{\mathrm{O}}$ algebras follows immediately from a simple and elegant theorem proven in [2]. Because the proof is short and presents a striking contrast with the arguments required for the other types, we repeat it here.

**Theorem 4.4.** *Let* $\underline{\mathbf{M}}$ *be a finite algebra. Assume there is a pair of binary algebraic operations* $\vee$ *and* $\wedge$ *on* $\underline{\mathbf{M}}$ *such that* $\langle M; \vee, \wedge\rangle$ *is a lattice. Then the structure* $\underset{\sim}{\mathbf{M}} := \langle M; \vee, \wedge, R_{2|M|}, \mathcal{T}\rangle$ *yields a duality on* $\mathbb{ISP}(\underline{\mathbf{M}})$.

**Proof.** Let $\mathbf{A}$ be a finite algebra in $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ and let $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ be a morphism. We can write $\alpha(\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})) = \{m_1, \dots, m_n\}$, for some $n \leqslant |M|$. The dual $\mathrm{D}(\mathbf{A})$ is a substructure of $\underset{\sim}{\mathbf{M}}^A$. So $\mathrm{D}(\mathbf{A})$ is finite and has a lattice reduct. This means that, for each $k \in \{1, \dots, n\}$, we can define $x_k := \bigwedge \alpha^{-1}(m_k)$ and $y_k := \bigvee \alpha^{-1}(m_k)$ in $\mathrm{D}(\mathbf{A})$. By Lemma 1.1, we know that $\alpha$ is given by evaluation at some $a \in A$ on the set $\{x_1, y_1, \dots, x_n, y_n\}$. Now consider any $z \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. There is some $k \in \{1, \dots, n\}$ such that $\alpha(z) = m_k$. We have $x_k \leqslant z \leqslant y_k$. As $\alpha$ preserves $\vee$ and $\wedge$, we also have $\alpha(x_k) = m_k = \alpha(y_k)$. So

$$m_k = \alpha(x_k) = x_k(a) \leqslant z(a) \leqslant y_k(a) = \alpha(y_k) = m_k\,.$$

This implies that $\alpha(z) = m_k = z(a)$, whence $\alpha$ is an evaluation. Thus $\underset{\sim}{\mathbf{M}}$ yields a duality on $\boldsymbol{\mathcal{A}}$, by the Duality Compactness Theorem 1.2. □

**Corollary 4.5.** *Let* $\underline{\mathbf{M}} = \langle M; F\rangle$ *be a finite unary algebra. Assume there is a total order on* $M$ *that is preserved by each map in* $F$. *Then* $\underline{\mathbf{M}}$ *is dualisable.*

There are several natural generalisations of Theorem 4.4 that do not hold. We know that a finite algebra is dualisable if it has a pair of algebraic lattice operations.

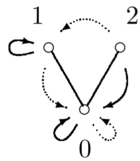


Fig. 3.

But an algebraic semilattice operation or an algebraic majority operation is not enough to guarantee the dualisability of an algebra. To see this, define the unary algebra $\underline{\mathbf{M}} := \langle\{0,1,2\}; 010, 001\rangle$. The maps 010 and 001 are both endomorphisms of the meet semilattice shown in Fig. 3. So there is an algebraic meet-semilattice operation on $\underline{\mathbf{M}}$. There is an algebraic majority operation $m : \underline{\mathbf{M}}^3 \to \underline{\mathbf{M}}$, given by

$$m(a,b,c) = \begin{cases} \mathrm{maj}(a,b,c) & \text{if } |\{a,b,c\}| \leqslant 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Nevertheless, the algebra $\underline{\mathbf{M}}$ is not dualisable. (This will follow from Lemma 5.3.)

There are some semilattices and median algebras whose endomorphisms do form dualisable unary algebras, as the following general argument shows. Let $\mathbf{M}_0$ be a finite algebra (of any type) such that $|\mathrm{Con}(\mathbf{M}_0)| \leqslant 3$ and let $F \subseteq \mathrm{End}(\mathbf{M}_0)$. Then $\langle M; F\rangle$ is a zero- or one-kernel algebra and is therefore dualisable, by Theorems 2.7 and 3.9. This argument can also be applied, for example, to the dihedral group $\mathbf{G}$ of order $2p$, for any prime $p$. Any set of endomorphisms of $\mathbf{G}$ is the set of operations of a dualisable unary algebra.

It follows straight from Corollary 4.5 that every type-$(2)_{\mathrm{O}}$ algebra is dualisable. In the next section, we shall show that every algebra of type $(2)_{\mathrm{P}}$ or $(2)_{\mathrm{M}}$ is non-dualisable. The remainder of this section is devoted to proving that the rest, all the algebras of type $(2)_{\mathrm{R}}$, are dualisable.

Let $\underline{\mathbf{M}}$ be a three-element unary algebra that has type $(2)_{\mathrm{R}}$. Within the quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$, we can restrict our attention to subalgebras of powers of $\underline{\mathbf{M}}$. Let $S$ be a non-empty set. For each $a \in M^S$, we say that

$$\mathcal{P}(a) := \{a^{-1}(0), a^{-1}(1), a^{-1}(2)\}\backslash\{\emptyset\}$$

is the **partition of $S$ determined by** $a$.

Now let $\mathbf{A}$ be a subalgebra of $\underline{\mathbf{M}}^S$. The structure of $\mathbf{A}$ may be quite complicated. However, we shall show that the homomorphisms in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ are all determined by their restrictions to a very simple subalgebra $\mathbf{A}_{\downarrow 2}$ of $\mathbf{A}$. The underlying set of $\mathbf{A}_{\downarrow 2}$ is given by

$$A_{\downarrow 2} := \{a \in A \mid |a(S)| \leqslant 2\}\,.$$

Since $\underline{\mathbf{M}}$ has type $(2)_{\mathrm{R}}$, we know that both $f_1 = 010$ and $f_2 = 002$ are term functions of $\underline{\mathbf{M}}$. This implies that $A_{\downarrow 2}$ is not empty. Indeed, we have $f_1(A) \cup f_2(A) \subseteq A_{\downarrow 2}$.

We shall be making constant use of the well-behaved term functions $f_1$ and $f_2$ of $\underline{\mathbf{M}}$. The maps $f_1$ and $f_2$ separate the elements of $M$. Moreover, for each $m \in \{1, 2\}$, the map $f_m$ on $M$ is idempotent with image $\{0, m\}$, and $f_m^{-1}(m) = \{m\}$.

**Lemma 4.6.** *Assume that* $\underline{\mathbf{M}}$ *has type* $(2)_{\mathrm{R}}$. *Let* $\mathbf{A}$ *be a subalgebra of* $\underline{\mathbf{M}}^S$, *for some non-empty set* $S$, *and let* $\mathbf{P}$ *be a petal of* $\mathbf{A}_{\downarrow 2}$. *Then all non-centre elements of* $\mathbf{P}$ *determine the same partition of* $S$.

**Proof.** Let $a \in P \backslash C_{\mathbf{A}}$ and let $u$ be a unary term function of $\underline{\mathbf{M}}$ with $u(a) \notin C_{\mathbf{A}}$. Once we have shown that $\mathcal{P}(a) = \mathcal{P}(u(a))$, the result will follow since the graph $G^*(\mathbf{P})$ is connected. Since $a \in A_{\downarrow 2}$, we have $|a(S)| \leqslant 2$. First assume that $a \in \{0, m\}^S$, for some $m \in \{1, 2\}$. Since $f_m$ fixes 0 and $m$, we have $u(f_m(a)) = u(a) \notin C_{\mathbf{A}}$. This implies that the term function $u \circ f_m$ of $\underline{\mathbf{M}}$ is not constant. So $u(0) \neq u(m)$, and therefore $\mathcal{P}(a) = \mathcal{P}(u(a))$. Now assume that $a \in \{1, 2\}^S$. As $u(a) \notin C_{\mathbf{A}}$, the term function $u$ of $\underline{\mathbf{M}}$ is not constant. Since $\underline{\mathbf{M}}$ does not have any term functions with kernel $\{0|12\}$, we have $u(1) \neq u(2)$. Thus $\mathcal{P}(a) = \mathcal{P}(u(a))$. □

The previous lemma tells us that every petal of $\mathbf{A}_{\downarrow 2}$ is isomorphic to a subalgebra of $\underline{\mathbf{M}}^2$. So, by Lemma 2.3, the subalgebra $\mathbf{A}_{\downarrow 2}$ of $\mathbf{A}$ is simply a coproduct of subalgebras of $\underline{\mathbf{M}}^2$. We will now show that homomorphisms from $\mathbf{A}$ to $\underline{\mathbf{M}}$ are determined by their restrictions to $\mathbf{A}_{\downarrow 2}$.

**Lemma 4.7.** *Assume that* $\underline{\mathbf{M}}$ *has type* $(2)_{\mathrm{R}}$. *Let* $\mathbf{A}$ *be a subalgebra of* $\underline{\mathbf{M}}^S$, *for some non-empty set* $S$, *and let* $x : \mathbf{A}_{\downarrow 2} \to \underline{\mathbf{M}}$ *be a homomorphism. Then* $x$ *has an extension to* $\mathbf{A}$ *if and only if* $x(f_1(a)) = 0$ *or* $x(f_2(a)) = 0$, *for all* $a \in A \backslash A_{\downarrow 2}$. *Furthermore, if* $x$ *has an extension to* $\mathbf{A}$, *then that extension is unique.*

**Proof.** First let $a \in A \backslash A_{\downarrow 2}$ and let $m \in \{1, 2\}$. Then $f_m(a) \in A_{\downarrow 2}$ and, since $f_m$ is idempotent, we have $x(f_m(a)) = x(f_m \circ f_m(a)) = f_m(x(f_m(a))) \in \{0, m\}$. We have shown that

$$x(f_m(a)) \in \{0, m\}, \text{ for all } a \in A \backslash A_{\downarrow 2} \text{ and each } m \in \{1, 2\}. \qquad (*)$$

Now assume that $x$ extends to a homomorphism $\bar{x} : \mathbf{A} \to \underline{\mathbf{M}}$. For all $a \in A \backslash A_{\downarrow 2}$ and $m \in \{1, 2\}$, we have

$$\bar{x}(a) = m \iff f_m(\bar{x}(a)) = m \iff x(f_m(a)) = m.$$

Therefore $\bar{x}$ is the unique extension of $x$ to $\mathbf{A}$. It also follows that $x(f_1(a)) \neq 1$ or $x(f_2(a)) \neq 2$, for all $a \in A \backslash A_{\downarrow 2}$. So, by $(*)$, we have $x(f_1(a)) = 0$ or $x(f_2(a)) = 0$, for all $a \in A \backslash A_{\downarrow 2}$.

Conversely, assume that $x(f_1(a)) = 0$ or $x(f_2(a)) = 0$, for every $a \in A \backslash A_{\downarrow 2}$. Using $(*)$, we can define the extension $\bar{x} : A \to M$ of the map $x$ so that, for every

Table 4. Proof of Lemma 4.7.

| $x(f_1(a))$ | $x(f_2(a))$ | $\bar{x}(a)$ | $x(f_1(\tilde{a}))$ | $x(f_2(\tilde{a}))$ | $\bar{x}(\tilde{a})$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 2 | 2 |
| 0 | 2 | 2 | 1 | 0 | 1 |

$a \in A \backslash A_{\downarrow 2}$, we have

$$\bar{x}(a) = \begin{cases} 0 & \text{if } x(f_1(a)) = x(f_2(a)) = 0\,, \\ 1 & \text{if } x(f_1(a)) = 1\,, \\ 2 & \text{if } x(f_2(a)) = 2\,. \end{cases}$$

We want to show that $\bar{x} : \mathbf{A} \to \underline{\mathbf{M}}$ is a homomorphism.

Let $a \in A \backslash A_{\downarrow 2}$. We will use the description of the unary term functions of $\underline{\mathbf{M}}$ given in Lemma 4.2. First, let $p, q \in M$ and assume that $ppq$ is a term function of $\underline{\mathbf{M}}$. Then, using Table 4, we see that

$$ppq(\bar{x}(a)) = ppq(x(f_2(a))) = x(ppq \circ 002(a)) = x(ppq(a)) = \bar{x}(ppq(a))\,.$$

Assume that $pqp$ is a term function of $\underline{\mathbf{M}}$. Then

$$pqp(\bar{x}(a)) = pqp(x(f_1(a))) = x(pqp \circ 010(a)) = x(pqp(a)) = \bar{x}(pqp(a))\,.$$

Lastly, assume that 021 is a term function of $\underline{\mathbf{M}}$, and define $\widetilde{a} := 021(a)$ in $A$. Then

$$x(f_1(\widetilde{a})) = x(010 \circ 021(a)) = x(001(a)) = x(021 \circ 002(a)) = 021(x(f_2(a)))$$

and

$$x(f_2(\widetilde{a})) = x(002 \circ 021(a)) = x(020(a)) = x(021 \circ 010(a)) = 021(x(f_1(a)))\,.$$

It now follows from Table 4 that $\bar{x}(021(a)) = \bar{x}(\widetilde{a}) = 021(\bar{x}(a))$. Thus $\bar{x}$ is a homomorphism. □

The subalgebra $\mathbf{A}$ of $\underline{\mathbf{M}}^S$ is **balanced** if every homomorphism from $\mathbf{A}$ to $\underline{\mathbf{M}}$ is the restriction of a projection. It is easy to check that every algebra in $\boldsymbol{\mathcal{A}}$ is isomorphic to a balanced algebra (see [1, Sec. 10.4.1]). Therefore we can further restrict our attention to those algebras in $\boldsymbol{\mathcal{A}}$ that are balanced.

So far, we have twice used a particular trick for helping to establish dualisability. Given a pair of homomorphisms in the dual of an algebra, we constructed a sequence of homomorphisms, from one to the other, so that homomorphisms that were adjacent in the sequence were nearly equal. (We did this in the proofs of both Lemmas 2.4 and 3.9.) We will be using the same trick to help prove that type-$(2)_{\mathrm{R}}$ algebras are dualisable.

Let $x, y \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with $x \neq y$. We say that the homomorphisms $x$ and $y$ are **almost equal** if there is a petal $\mathbf{P}$ of $\mathbf{A}_{\downarrow 2}$ such that $x$ and $y$ agree on $A_{\downarrow 2} \backslash P$.

**Lemma 4.8.** *Assume that* $\underline{\mathbf{M}}$ *has type* $(2)_{\mathrm{R}}$ *and define* $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$*. Let* $\mathbf{A}$ *be a balanced subalgebra of* $\underline{\mathbf{M}}^S$*, for some non-empty finite set* $S$*, and let* $x, y \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ *with* $x \neq y$*. Then there is a sequence* $x = x_0, x_1, \dots, x_n = y$ *in* $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$*, for some* $n \in \mathbb{N}$*, such that* $x_i$ *is almost equal to* $x_{i+1}$ *and* $A_{\downarrow 2} \cap \mathrm{eq}(x, y) \subseteq A_{\downarrow 2} \cap \mathrm{eq}(x_i, y)$*, for every* $i \in \{0, \dots, n-1\}$*.*

**Proof.** Define $x_0 := x$. Now let $i \in \mathbb{N} \cup \{0\}$ and assume that the homomorphism $x_i$ in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ has already been defined, with $x_i \neq y$ and $A_{\downarrow 2} \cap \mathrm{eq}(x, y) \subseteq A_{\downarrow 2} \cap \mathrm{eq}(x_i, y)$.

To each petal of $\mathbf{A}_{\downarrow 2}$, we shall associate a subset of $S$. Let $\mathbf{P}$ be a petal of $\mathbf{A}_{\downarrow 2}$ and, for some $a \in P \backslash C_{\mathbf{A}}$, define the subset $S_{\mathbf{P}} := a^{-1}(x_i(a))$ of $S$. Since $\mathbf{A}$ is balanced, the homomorphism $x_i$ is the restriction of a projection. By Lemma 4.6, all non-centre elements of $\mathbf{P}$ determine the same partition of $S$. This implies that the subset $S_{\mathbf{P}}$ of $S$ is independent of our choice for $a$.

The algebra $\mathbf{A}_{\downarrow 2}$ is the coproduct of its petals. We want to find a petal $\mathbf{P}$ of $\mathbf{A}_{\downarrow 2}$ such that $x_i{\restriction}_P \neq y{\restriction}_P$ and the homomorphism $x_i{\restriction}_{A_{\downarrow 2} \backslash P} \cup y{\restriction}_P : \mathbf{A}_{\downarrow 2} \to \underline{\mathbf{M}}$ extends to a homomorphism in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. We are assuming that $x_i \neq y$. So, by Lemma 4.7, the homomorphisms $x_i$ and $y$ do not agree on $\mathbf{A}_{\downarrow 2}$. There must be a petal $\mathbf{Q}$ of $\mathbf{A}_{\downarrow 2}$ for which $x_i{\restriction}_Q \neq y{\restriction}_Q$. Assume that $x_i{\restriction}_{A_{\downarrow 2} \backslash Q} \cup y{\restriction}_Q : \mathbf{A}_{\downarrow 2} \to \underline{\mathbf{M}}$ does not extend to a homomorphism in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. We will show that there is a petal $\mathbf{R}$ of $\mathbf{A}_{\downarrow 2}$ such that $x_i{\restriction}_R \neq y{\restriction}_R$ and $S_{\mathbf{R}}$ is a proper subset of $S_{\mathbf{Q}}$.

Since $x_i{\restriction}_{A_{\downarrow 2} \backslash Q} \cup y{\restriction}_Q : \mathbf{A}_{\downarrow 2} \to \underline{\mathbf{M}}$ does not extend to $\mathbf{A}$, it follows by Lemma 4.7 that there exists $a \in A \backslash A_{\downarrow 2}$ and $\{k, \ell\} = \{1, 2\}$, with $f_k(a) \in Q$ and $f_\ell(a) \in A_{\downarrow 2} \backslash Q$, such that $y(f_k(a)) \neq 0 \neq x_i(f_\ell(a))$. Since $x_i(f_\ell(a)) \neq 0$ and $x_i$ is the restriction of a projection, we get $x_i(f_\ell(a)) = \ell$. Using Lemma 4.7 again, we must have $x_i(f_k(a)) = 0$ and $y(f_\ell(a)) = 0$. Since $|a(S)| = 3$, we know that $f_k(a), f_\ell(a) \notin C_{\mathbf{A}}$. So define $\mathbf{R}$ to be the petal of $\mathbf{A}_{\downarrow 2}$ containing $f_\ell(a)$. Then $x_i(f_\ell(a)) = \ell \neq 0 = y(f_\ell(a))$, and therefore $x_i{\restriction}_R \neq y{\restriction}_R$. We have

$$S_{\mathbf{R}} = f_\ell(a)^{-1}(\ell) = a^{-1}(\ell) \subset a^{-1}(0) \cup a^{-1}(\ell) = f_k(a)^{-1}(0) = S_{\mathbf{Q}} \,.$$

Note that $S_{\mathbf{R}}$ is a proper subset of $S_{\mathbf{Q}}$, as $|a(S)| = 3$ and therefore $a^{-1}(0) \neq \emptyset$. We have constructed the petal $\mathbf{R}$ as claimed.

Using the above argument and the finiteness of $S$, we can choose a petal $\mathbf{P}$ of $\mathbf{A}_{\downarrow 2}$ so that $x_i{\restriction}_P \neq y{\restriction}_P$ and $x_i{\restriction}_{A_{\downarrow 2} \backslash P} \cup y{\restriction}_P : \mathbf{A}_{\downarrow 2} \to \underline{\mathbf{M}}$ extends to a homomorphism in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. Call this extension $x_{i+1}$. The homomorphisms $x_i$ and $x_{i+1}$ are almost equal, and $A_{\downarrow 2} \cap \mathrm{eq}(x_i, y)$ is a proper subset of $A_{\downarrow 2} \cap \mathrm{eq}(x_{i+1}, y)$. Since $\mathbf{A}$ is finite, there will be some $n \in \mathbb{N}$ such that $A_{\downarrow 2} \subseteq \mathrm{eq}(x_n, y)$. By Lemma 4.7, it follows that $x_n = y$. So the claim holds. □

The next lemma completes the preparation for our proof that type-$(2)_{\mathrm{R}}$ algebras are dualisable. This lemma is the only place we use the fact that algebras with type $(2)_{\mathrm{R}}$ do not have type $(2)_{\mathrm{M}}$.

**Lemma 4.9.** *Assume that $\underline{\mathbf{M}}$ has type $(2)_{\mathrm{R}}$. Let $\mathbf{A}$ be a subalgebra of $\underline{\mathbf{M}}^S$, for some non-empty set $S$, and let $m \in M$. If $A$ does not contain the constant map in $M^S$ with value $m$, then there is a homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$ such that $m \notin x(A)$.*

**Proof.** Let $F$ be the set of unary term functions of $\underline{\mathbf{M}}$. For each $m \in M$, let $\widehat{m}$ be the constant map in $M^S$ with value $m$. We must have $\widehat{0} \in A$, since $000 = f_2 \circ f_1 \in F$. So, by symmetry, we can assume that $\widehat{2} \notin A$. We want to find a homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$ such that $2 \notin x(A)$.

Since $\widehat{2} \notin A$, we must have $222 \notin F$. First, assume that $111 \notin F$. For all $u \in F$, we have $u \circ 000 \in F$, and therefore $u(0) = 0$. So the constant map $\underline{0} : \mathbf{A} \to \underline{\mathbf{M}}$ is a homomorphism, and $2 \notin \underline{0}(A)$.

Now assume that $111 \in F$. For all $u \in F$, we have $u \circ 000 \in F$ and $u \circ 111 \in F$. Since $222 \notin F$, we must have $u(0) \in \{0, 1\}$ and $u(1) \in \{0, 1\}$, for all $u \in F$. Using Lemma 4.2, it follows that

$$F \subseteq \{012, 010, 101\} \cup \{00q, 11q \mid q \in M\}.$$

Since $010 = f_1 \in F$ and $\underline{\mathbf{M}}$ does not have type $(2)_{\mathrm{M}}$, we also know that $\{001, 110\} \nsubseteq F$. There are three cases to consider.

**Case 1.** $001 \in F$ and $110 \notin F$. We have $101 \notin F$ and $112 \notin F$, since $101 \circ 001 = 010 \circ 112 = 110 \notin F$. So $F \subseteq \{012, 010, 111\} \cup \{00q \mid q \in M\}$. Since $\widehat{2} \notin A$, there is a homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$, given by $x := \underline{0}{\restriction}_{A \setminus \{\widehat{1}\}} \cup \underline{1}{\restriction}_{\{\widehat{1}\}}$, with $2 \notin x(A)$.

**Case 2.** $110 \in F$ and $001 \notin F$. This gives us $101 \notin F$, as $101 \circ 110 = 001 \notin F$. Therefore $F \subseteq \{012, 010, 000, 002\} \cup \{11q \mid q \in M\}$. Since $f_2$ fixes 0 and 2, we have $f_2(A) = A \cap \{0, 2\}^S$. As $\widehat{2} \notin A$, there is a homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$, given by $x := \underline{1}{\restriction}_{A \setminus f_2(A)} \cup \underline{0}{\restriction}_{f_2(A)}$, with $2 \notin x(A)$.

**Case 3.** $001 \notin F$ and $110 \notin F$. We have $112 \notin F$, since $010 \circ 112 = 110 \notin F$. So $F \subseteq \{012, 010, 101, 000, 002, 111\}$. Choose any $s \in S$. Then we can define the homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$ by $x := f_1 \circ \pi_s{\restriction}_{A \setminus f_2(A)} \cup \underline{0}{\restriction}_{f_2(A)}$, and $2 \notin x(A)$. □

**Theorem 4.10.** *Every three-element unary algebra with type $(2)_{\mathrm{R}}$ is dualisable.*

**Proof.** Assume $\underline{\mathbf{M}}$ has type $(2)_{\mathrm{R}}$ and define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$. Define the alter ego $\underset{\sim}{\mathbf{M}} := \langle \{0, 1, 2\}; R_8, \mathcal{T} \rangle$ of $\underline{\mathbf{M}}$. (By doing a little extra work at one of the steps in this proof, we can actually get by with $R_6$ instead of $R_8$.) Let $\mathbf{A}$ be a balanced subalgebra of $\underline{\mathbf{M}}^S$, for some non-empty finite set $S$, and let $\alpha : \mathrm{D}(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ be a morphism. We will show that $\alpha$ is an evaluation. It will then follow by the Duality Compactness Theorem 1.2 that $\underset{\sim}{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$.

First assume that $\alpha$ is constant. Let $m$ be the value of $\alpha$ in $M$, and suppose that $A$ does not contain the constant map $\widehat{m}$ in $M^S$ with value $m$. By Lemma 4.9, there is some $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ such that $m \notin x(A)$. The set $x(A)$ is a unary algebraic relation on $\underline{\mathbf{M}}$. Since $\alpha$ preserves $x(A)$, we have $\alpha(x) \in x(A)$. So $\alpha(x) \neq m$, which is a contradiction. We have shown that $\widehat{m} \in A$. The map $\alpha$ is given by evaluation at $\widehat{m}$, since every element of $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ is the restriction of a projection.

Now assume that the map $\alpha$ is not constant. There are $v_1, v_2 \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ such that $\alpha(v_1) \neq \alpha(v_2)$. By Lemma 4.8, there is a sequence $v_1 = v_{10}, v_{11}, \dots, v_{1n} = v_2$ in $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$, for some $n \in \mathbb{N}$, such that $v_{1i}$ is almost equal to $v_{1i+1}$, for all $i \in \{0, \dots, n-1\}$. Since $\alpha(v_1) \neq \alpha(v_2)$, there is some $j \in \{0, \dots, n-1\}$ with $\alpha(v_{1j}) \neq \alpha(v_{1j+1})$. Define $y_1 := v_{1j}$ and $y_2 := v_{1j+1}$. Since $y_1$ and $y_2$ are almost equal, there is a petal $\mathbf{P}_y$ of $\mathbf{A}_{\downarrow 2}$ such that $A_{\downarrow 2} \backslash P_y \subseteq \mathrm{eq}(y_1, y_2)$. As $y_1 \neq y_2$, we must have $y_1{\restriction}_{A_{\downarrow 2}} \neq y_2{\restriction}_{A_{\downarrow 2}}$, by Lemma 4.7. So $y_1{\restriction}_{P_y} \neq y_2{\restriction}_{P_y}$.

**Case 1.** $P_y$ is a support for $\alpha$. We begin by showing that there is some $a \in P_y$ such that $\alpha$ is given by evaluation at $a$ on $\{y_1, y_2\}$. Since $\alpha$ preserves $R_8$, we can use Lemma 1.1 to find some $b \in A$ such that $\alpha$ is given by evaluation at $b$ on $\{y_1, y_2\}$. Assume that $b \notin P_y$. We have $y_1(b) = \alpha(y_1) \neq \alpha(y_2) = y_2(b)$. As $A_{\downarrow 2} \backslash P_y \subseteq \mathrm{eq}(y_1, y_2)$, this implies that $b \notin A_{\downarrow 2}$ and therefore $|b(S)| = 3$. Since $f_1$ and $f_2$ separate the elements of $M$, we have $f_k(y_1(b)) \neq f_k(y_2(b))$, for some $k \in \{1, 2\}$. As $y_1(f_k(b)) \neq y_2(f_k(b))$ and $A_{\downarrow 2} \backslash P_y \subseteq \mathrm{eq}(y_1, y_2)$, it follows that $f_k(b) \in P_y$. We shall show that $\alpha$ is given by evaluation at $f_k(b)$ on $\{y_1, y_2\}$. As $|b(S)| = 3$, the elements $f_1(b)$ and $f_2(b)$ of $\mathbf{A}$ determine two different two-block partitions of $S$. So $f_1(b)$ and $f_2(b)$ belong to different petals of $\mathbf{A}_{\downarrow 2}$, by Lemma 4.6. Choose $\ell \in \{1, 2\}$ with $\ell \neq k$. Then $f_\ell(b) \notin P_y$ and so

$$f_\ell(y_1(b)) = y_1(f_\ell(b)) = y_2(f_\ell(b)) = f_\ell(y_2(b)),$$

as $A_{\downarrow 2} \backslash P_y \subseteq \mathrm{eq}(y_1, y_2)$. Since $y_1(b) \neq y_2(b)$, this gives us $\{y_1(b), y_2(b)\} = \{0, k\}$. Thus $\alpha(y_i) = y_i(b) = f_k(y_i(b)) = y_i(f_k(b))$, for each $i \in \{1, 2\}$.

We have shown that $\alpha$ is given by evaluation at some $a \in P_y$ on $\{y_1, y_2\}$. By Lemma 4.6, all non-centre elements of $\mathbf{P}_y$ determine the same partition of $S$. Since $P_y \subseteq A_{\downarrow 2}$, this partition has at most two blocks. So there are at most two functions from $P_y$ to $M$ that are the restriction of a projection. Now let $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. As $\mathbf{A}$ is balanced, the homomorphisms $x$, $y_1$ and $y_2$ are restrictions of projections. Since $y_1{\restriction}_{P_y} \neq y_2{\restriction}_{P_y}$, there is some $i \in \{1, 2\}$ such that $x{\restriction}_{P_y} = y_i{\restriction}_{P_y}$. Therefore $\alpha(x) = \alpha(y_i) = y_i(a) = x(a)$, as $P_y$ is a support for $\alpha$. Thus $\alpha$ is an evaluation.

**Case 2.** $P_y$ is not a support for $\alpha$. By Lemma 4.8, there exist almost equal homomorphisms $z_1, z_2 \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ such that $z_1{\restriction}_{P_y} = z_2{\restriction}_{P_y}$ and $\alpha(z_1) \neq \alpha(z_2)$. There is a petal $\mathbf{P}_z$ of $\mathbf{A}_{\downarrow 2}$ with $A_{\downarrow 2} \backslash P_z \subseteq \mathrm{eq}(z_1, z_2)$. Using Lemma 4.7, we must have $z_1{\restriction}_{P_z} \neq z_2{\restriction}_{P_z}$. So $\mathbf{P}_y$ and $\mathbf{P}_z$ are different petals of $\mathbf{A}_{\downarrow 2}$.

The petals $\mathbf{P}_y$ and $\mathbf{P}_z$ determine two partitions of $S$ each with at most two blocks. So there are at most four functions from $P_y \cup P_z$ to $M$ that are the restriction of a projection. Since $\mathbf{A}$ is balanced, there is a subset $W$ of $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with $|W| \leqslant 7$ such that

$$\{y_1, y_2, z_1, z_2\} \subseteq W \quad \text{and} \quad \{w{\restriction}_{P_y \cup P_z} \mid w \in W\} = \{x{\restriction}_{P_y \cup P_z} \mid x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})\}.$$

(We can actually choose $W$ so that $|W| \leqslant 5$.) Define

$$A_W := \{a \in A \mid \alpha{\restriction}_W = e_{\mathbf{A}}(a){\restriction}_W\}\,.$$

Then $A_W$ is non-empty, by Lemma 1.1, as $\alpha$ preserves $R_8$.

We want to show that $A_W$ has only one element. To do this, let $a_1, a_2 \in A_W$, let $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ and let $i \in \{1, 2\}$. There is some $w \in W$ with $x{\restriction}_{P_y \cup P_z} = w{\restriction}_{P_y \cup P_z}$. Since $a_i \in A_W$, the map $\alpha$ is given by evaluation at $a_i$ on $W$. Therefore

$$y_1(a_i) = \alpha(y_1) \neq \alpha(y_2) = y_2(a_i) \quad \text{and} \quad z_1(a_i) = \alpha(z_1) \neq \alpha(z_2) = z_2(a_i)\,.$$

The maps $f_1$ and $f_2$ separate the elements of $M$. So there is some $j \in \{1, 2\}$ with

$$y_1(f_j(a_i)) = f_j(y_1(a_i)) \neq f_j(y_2(a_i)) = y_2(f_j(a_i))\,.$$

As $A_{\downarrow 2} \backslash P_y \subseteq \mathrm{eq}(y_1, y_2)$, this implies that $f_1(a_i) \in P_y \backslash C_{\mathbf{A}}$ or $f_2(a_i) \in P_y \backslash C_{\mathbf{A}}$. Similarly, we have $f_1(a_i) \in P_z \backslash C_{\mathbf{A}}$ or $f_2(a_i) \in P_z \backslash C_{\mathbf{A}}$. Thus $f_1(a_i), f_2(a_i) \in P_y \cup P_z$, as $\mathbf{P}_y$ and $\mathbf{P}_z$ are distinct petals of $\mathbf{A}_{\downarrow 2}$. Since $x{\restriction}_{P_y \cup P_z} = w{\restriction}_{P_y \cup P_z}$, this gives us $f_1(x(a_i)) = f_1(w(a_i))$ and $f_2(x(a_i)) = f_2(w(a_i))$. As $f_1$ and $f_2$ separate the elements of $M$, we have

$$x(a_1) = w(a_1) = \alpha(w) = w(a_2) = x(a_2)\,.$$

Since $\mathbf{A}$ is separated by homomorphisms into $\underline{\mathbf{M}}$, it follows that $a_1 = a_2$.

Now let $a$ be the unique element of $A_W$. To see that $\alpha$ is given by evaluation at $a$, let $x \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. Since $\alpha$ preserves $R_8$, there is some $b \in A$ such that $\alpha$ is given by evaluation at $b$ on the set $W \cup \{x\}$. As $a$ is the only element of $A_W$, we must have $b = a$ and therefore $\alpha(x) = x(a)$. Thus $\alpha$ is an evaluation. □

## 5. Non-Dualisable Three-Element Unary Algebras

Perhaps surprisingly, proofs of non-dualisability are often easier than proofs of dualisability. The ghost-element method provides an extremely elegant way to show that a finite algebra is not dualisable. Constructing a ghost-element proof can require a fair amount of inspiration. However, verifying that the construction is correct often involves only routine calculations. The ghost-element method, which was introduced in [8], has been applied extensively [1, 3, 4, 5, 7, 12]. We now use this method to prove that the remaining three-element unary algebras are non-dualisable.

The following lemma (based on [1, Sec. 10.5.1]) provides a basic description of the ghost-element method. For a subalgebra $\mathbf{A}$ of $\underline{\mathbf{M}}^S$ and $s \in S$, let $\rho_s := \pi_s{\restriction}_A : \mathbf{A} \to \underline{\mathbf{M}}$ denote the natural projection homomorphism.

**Lemma 5.1.** *Let $\underline{\mathbf{M}}$ be a finite algebra and define $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$. Let $\mathbf{A}$ be a subalgebra of $\underline{\mathbf{M}}^S$, for some set $S$, and let $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$. Assume that*

(i) *$\alpha$ has a finite support in $A$, and*
(ii) *$\alpha$ agrees with an evaluation on each finite subset of $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$.*

*Define $g_\alpha \in M^S$ by $g_\alpha(s) := \alpha(\rho_s)$, for all $s \in S$. If $\underline{\mathbf{M}}$ is dualisable, then $g_\alpha \in A$.*

We can show that a finite algebra $\underline{\mathbf{M}}$ is not dualisable by finding an algebra $\mathbf{A}$ and a map $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$, satisfying the conditions of Lemma 5.1, such that $g_\alpha \notin A$. The element $g_\alpha$ is then called a **ghost element of A**. For all our ghost-element proofs, we will use the following refinement of Lemma 5.1. (The result below is the unpublished precursor to the Inherently Non-dualisable Algebra Theorem [1, 5]. The proof is included here for the sake of completeness. We cannot use the Inherently Non-dualisable Algebra Theorem itself, since there are no inherently non-dualisable unary algebras, by [2].)

**Lemma 5.2.** *Let* $\underline{\mathbf{M}}$ *be a finite algebra. Let* $\mathbf{A}$ *be a subalgebra of* $\underline{\mathbf{M}}^S$, *for some set* $S$, *and let* $A_0$ *be an infinite subset of* $A$. *Assume that, for each homomorphism* $x : \mathbf{A} \to \underline{\mathbf{M}}$, *the equivalence relation* $\ker(x{\restriction}_{A_0})$ *has a unique non-trivial block. Define* $g \in M^S$ *by* $g(s) := \rho_s(a_s)$, *where* $a_s$ *is any element of the non-trivial block of* $\ker(\rho_s{\restriction}_{A_0})$. *If* $\underline{\mathbf{M}}$ *is dualisable, then* $g \in A$.

**Proof.** Let $\boldsymbol{\mathcal{A}} := \mathbb{ISP}(\underline{\mathbf{M}})$ and define the map $\alpha : \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}}) \to M$ by $\alpha(x) := x(a_x)$, where $a_x$ is any member of the non-trivial block of $\ker(x{\restriction}_{A_0})$. Then $g(s) = \rho_s(a_s) = \alpha(\rho_s) = g_\alpha(s)$, for all $s \in S$, and therefore $g = g_\alpha$. By Lemma 5.1, it is now enough to prove that $\alpha$ has a finite support and that $\alpha$ agrees with an evaluation on each finite subset of $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$.

Choose any finite subset $B$ of $A_0$ such that $|B| \geqslant |M| + 1$. To see that $B$ is a support for $\alpha$, let $x, y \in \boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$ with $x{\restriction}_B = y{\restriction}_B$. The equivalence relation $\ker(x{\restriction}_B)$ on $B$ has at most $|M|$ blocks. Since $|B| > |M|$, there is a non-trivial block of $\ker(x{\restriction}_B)$. Choose some $b \in B$ that lies in this non-trivial block of $\ker(x{\restriction}_B) = \ker(y{\restriction}_B)$. Then $b$ belongs to the unique non-trivial block of $\ker(x{\restriction}_{A_0})$ and to the unique non-trivial block of $\ker(y{\restriction}_{A_0})$. So $\alpha(x) = x(b) = y(b) = \alpha(y)$. Thus $B$ is a finite support for $\alpha$.

Now let $X$ be a finite subset of $\boldsymbol{\mathcal{A}}(\mathbf{A}, \underline{\mathbf{M}})$. For each $x \in X$, let $A_x$ denote the unique non-trivial block of $\ker(x{\restriction}_{A_0})$. For each $x \in X$, the set $A_x$ is cofinite in $A_0$, since $\ker(x{\restriction}_{A_0})$ has finitely many blocks. So there is some $a \in \bigcap\{A_x \mid x \in X\}$. We have $\alpha(x) = x(a)$, for all $x \in X$. Thus $\alpha$ agrees with an evaluation on $X$. □

The next three results will complete the non-dualisability part of our characterisation. In the proofs, we will need to specify sequences in $\{0,1,2\}^{\mathbb{N}}$. Let $k, n_1, \dots, n_k \in \mathbb{N}$ and let $a, b_1, \dots, b_k \in \{0,1,2\}$. We shall define $a^{b_1 \dots b_k}_{n_1 \dots n_k} \in \{0,1,2\}^{\mathbb{N}}$ by

$$a^{b_1 \dots b_k}_{n_1 \dots n_k}(i) = \begin{cases} b_j & \text{if } i = n_j, \text{ for some } j \in \{1, \dots, k\}, \\ a & \text{otherwise}, \end{cases}$$

for all $i \in \mathbb{N}$.

The next theorem tells us that all the algebras with type $(2)_{\mathrm{P}}$ are non-dualisable.

**Theorem 5.3.** *Let* $\underline{\mathbf{M}}$ *be a dualisable unary algebra on the set* $\{0,1,2\}$. *If* $ppq$ *and* $pqp$ *are term functions of* $\underline{\mathbf{M}}$, *for some* $p, q \in \{0,1,2\}$ *with* $p \neq q$, *then so are* 010 *and* 002.

**Proof.** We will prove that, if both *ppq* and *pqp* are term functions of $\underline{\mathbf{M}}$, for some $p, q \in M$ with $p \neq q$, then 002 is as well. The rest of the result will then follow using conjugation by 021.

Assume that *ppq* and *pqp* are term functions of $\underline{\mathbf{M}}$, for some $p, q \in M$ with $p \neq q$. Define two subsets of $M^{\mathbb{N}}$ by

$$A_0 := \{0^{21}_{1n} \mid n \in \mathbb{N}\backslash\{1\}\} \quad \text{and} \quad B := \{0^{2\,1}_{mn} \mid m, n \in \mathbb{N}\backslash\{1\} \text{ and } m \neq n\}\,.$$

Let $\mathbf{A}$ denote the subalgebra of $\underline{\mathbf{M}}^{\mathbb{N}}$ generated by $A_0 \cup B$. Now choose a homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$. We shall show that $\ker(x{\restriction}_{A_0})$ has a unique non-trivial block.

**Case 1.** $2 \in x(A_0)$. There is some $n \in \mathbb{N}\backslash\{1\}$ with $x(0^{21}_{1n}) = 2$. Let $m \in \mathbb{N}\backslash\{1\}$. Then

$$0^{21}_{1n} \xrightarrow{ppq} p^{q}_{1} \xleftarrow{ppq} 0^{2\,1}_{1m}$$

in $\mathbf{A}$. Applying the homomorphism $x$ gives us

$$\boxed{2} \xrightarrow{ppq} q \xleftarrow{ppq} x(0^{2\,1}_{1m})$$

in $\underline{\mathbf{M}}$. (The box around 2 indicates that $x(0^{21}_{1n}) = 2$ by assumption.) Since $p \neq q$, it follows that $x(0^{2\,1}_{1m}) = 2$. So $x(A_0) = \{2\}$, and therefore $\ker(x{\restriction}_{A_0})$ has only one block.

**Case 2.** $x(A_0) \subseteq \{0, 1\}$. We can assume that $x(A_0) \neq \{0\}$. So there is some $n \in \mathbb{N}\backslash\{1\}$ such that $x(0^{21}_{1n}) = 1$. Let $m \in \mathbb{N}\backslash\{1, n\}$. Then

$$0^{21}_{1n} \xrightarrow{pqp} p^{q}_{n} \xleftarrow{pqp} 0^{2\,1}_{mn} \xrightarrow{ppq} p^{q}_{m} \xleftarrow{pqp} 0^{2\,1}_{1m}$$

in $\mathbf{A}$. Under the homomorphism $x$, we have

$$\boxed{1} \xrightarrow{pqp} q \xleftarrow{pqp} 1 \xrightarrow{ppq} p \xleftarrow{pqp} x(0^{2\,1}_{1m})$$

in $\underline{\mathbf{M}}$. Since $x(0^{2\,1}_{1m}) \neq 2$, this implies that $x(0^{2\,1}_{1m}) = 0$. So $A_0\backslash\{0^{21}_{1n}\}$ is the unique non-trivial block of $\ker(x{\restriction}_{A_0})$.

We have proven that $\ker(x{\restriction}_{A_0})$ has a unique non-trivial block, for each homomorphism $x : \mathbf{A} \to \underline{\mathbf{M}}$. Now define $g \in M^{\mathbb{N}}$ by $g(n) := \rho_n(a_n)$, where $a_n$ is chosen from the non-trivial block of $\ker(\rho_n{\restriction}_{A_0})$. The only block of $\ker(\rho_1{\restriction}_{A_0})$ is $A_0$. So $g(1) = \rho_1(0^{21}_{12}) = 2$. For each $n \in \mathbb{N}\backslash\{1\}$, the unique non-trivial block of $\ker(\rho_n{\restriction}_{A_0})$ is $A_0\backslash\{0^{21}_{1n}\}$, and therefore $g(n) = \rho_n(0^{2\;\;1}_{1\,n+1}) = 0$. Thus $g = 0^2_1$. We are assuming that $\underline{\mathbf{M}}$ is dualisable and so, by Lemma 5.2, we must have $g \in A$. Therefore $0^2_1 \in \mathrm{sg}_{\underline{\mathbf{M}}^{\mathbb{N}}}(A_0 \cup B)$, whence 002 must be a term function of $\underline{\mathbf{M}}$. $\square$

Now we will show that each algebra with type $(2)_{\mathrm{M}}$ is non-dualisable.

**Theorem 5.4.** *Let* $\underline{\mathbf{M}}$ *be a dualisable unary algebra on the set* $\{0, 1, 2\}$.

(i) *If* 010, 001 *and* 110 *are term functions of* $\underline{\mathbf{M}}$, *then so is* 222.
(ii) *If* 002, 020 *and* 202 *are term functions of* $\underline{\mathbf{M}}$, *then so is* 111.

**Proof.** We will prove (i). Claim (ii) will then follow using conjugation by 021. Assume that 010, 001 and 110 are term functions of $\underline{\mathbf{M}}$. Define two subsets of $M^{\mathbb{N}}$ by

$$A_0 := \{2^{0\ \ 1}_{k\,k+1} \mid k \in 2\mathbb{N}\} \quad \text{and} \quad B := \{0^{1\,1\,2\,2}_{k\ell mn} \mid k, \ell, m, n \in \mathbb{N} \text{ are distinct}\}\,.$$

Let $\mathbf{A}$ denote the subalgebra of $\underline{\mathbf{M}}^{\mathbb{N}}$ generated by $A_0 \cup B$.

Let $x : \mathbf{A} \to \underline{\mathbf{M}}$ be a homomorphism. We want to show that $\ker(x{\restriction}_{A_0})$ has a unique non-trivial block. So we can assume that $x(A_0) \neq \{2\}$. There is some $k \in 2\mathbb{N}$ such that $x(2^{0\ \ 1}_{k\,k+1}) \in \{0, 1\}$. Let $\ell \in 2\mathbb{N}\backslash\{k\}$. Then $k$, $k+1$, $\ell$ and $\ell+1$ are all distinct, and

$$\begin{array}{ccccccccc} 2^{0\ \ 1}_{k\,k+1} & \xrightarrow{110} & 0^{1\ \ 1}_{k\,k+1} & \xleftarrow{010} & 0^{1\ \ 1\ \ 2\ \ 2}_{k\,k+1\,\ell\,\ell+1} & \xrightarrow{001} & 0^{1\ \ 1}_{\ell\,\ell+1} & \xleftarrow{110} & 2^{0\ \ 1}_{\ell\,\ell+1} \\ & & & & \Downarrow x & & & & \\ \boxed{0,1} & \xrightarrow{110} & 1 & \xleftarrow{010} & 1 & \xrightarrow{001} & 0 & \xleftarrow{110} & 2\,. \end{array}$$

So $x(2^{0\ \ 1}_{\ell\,\ell+1}) = 2$, whence $A_0\backslash\{2^{0\ \ 1}_{k\,k+1}\}$ is the unique non-trivial block of $\ker(x{\restriction}_{A_0})$.

Define $g \in M^{\mathbb{N}}$ by $g(n) := \rho_n(a_n)$, where $a_n$ is any member of the non-trivial block of $\ker(\rho_n{\restriction}_{A_0})$. Then $g$ is the constant sequence $\widehat{2}$. By Lemma 5.2, we must have $g \in A$. Therefore $\widehat{2} \in \mathrm{sg}_{\underline{\mathbf{M}}^{\mathbb{N}}}(A_0 \cup B)$, whence 222 is a term function of $\underline{\mathbf{M}}$. □

The last claim we need to prove is that every three-element unary algebra with three kernels is non-dualisable. We shall obtain this as a corollary of the following stronger result.

**Theorem 5.5.** *Let $\underline{\mathbf{M}}$ be a finite unary algebra with at least three elements. Assume that, for each $m \in M$, the equivalence relation coming from the two-block partition $\{\{m\}, M\backslash\{m\}\}$ is a kernel of $\underline{\mathbf{M}}$. Then $\underline{\mathbf{M}}$ is not dualisable.*

**Proof.** We can assume that $M = \{0, \ldots, n\}$, for some $n \in \mathbb{N}$ with $n \geqslant 2$. For each $m \in M$, there is a unary term function $u_m$ of $\underline{\mathbf{M}}$ whose kernel is the equivalence relation coming from $\{\{m\}, M\backslash\{m\}\}$. Define two subsets of $M^{\mathbb{N}}$ by

$$A_0 := \{0^{11}_{1k} \mid k \in \mathbb{N}\backslash\{1\}\} \quad \text{and} \quad B := \{0^{112}_{1k\ell} \mid k, \ell \in \mathbb{N}\backslash\{1\} \text{ and } k \neq \ell\}\,.$$

Let $\mathbf{A}$ be the subalgebra of $\underline{\mathbf{M}}^{\mathbb{N}}$ generated by $A_0 \cup B$, and let $x : \mathbf{A} \to \underline{\mathbf{M}}$ be a homomorphism. We will show that $\ker(x{\restriction}_{A_0})$ has a unique non-trivial block.

**Case 1.** $m \in x(B)$, for some $m \in M\backslash\{0, 1, 2\}$. There exist $k, \ell \in \mathbb{N}\backslash\{1\}$, with $k \neq \ell$, such that $x(0^{112}_{1k\ell}) = m$. Let $j \in \mathbb{N}\backslash\{1\}$. Then

$$0^{112}_{1k\ell} \xrightarrow{u_m} * \xleftarrow{u_m} 0^{11}_{1j} \overset{x}{\Longrightarrow} \boxed{\mathrm{m}} \xrightarrow{u_m} * \xleftarrow{u_m} x(0^{11}_{1j})\,.$$

(Here we are using $*$ as a wild card.) This implies that $x(0^{11}_{1j}) = m$. Therefore $x(A_0) = \{m\}$, and $\ker(x{\restriction}_{A_0})$ has only one block.

**Case 2.** $m \in x(A_0)$, for some $m \in M \backslash \{0,1\}$. There is some $k \in \mathbb{N} \backslash \{1\}$ such that $x(0^{11}_{1k}) = m$. For all $j \in \mathbb{N} \backslash \{1\}$, we have

$$0^{11}_{1k} \xrightarrow{u_m} * \xleftarrow{u_m} 0^{11}_{1j} \xRightarrow{x} \boxed{\mathrm{m}} \xrightarrow{u_m} * \xleftarrow{u_m} x(0^{11}_{1j})\,,$$

and therefore $x(0^{11}_{1j}) = m$. So $x(A_0) = \{m\}$.

**Case 3.** $x(A_0) \subseteq \{0,1\}$ and $x(B) \subseteq \{0,1,2\}$. We can assume that $x{\restriction}_{A_0}$ is not constant. So there exist $k, \ell \in \mathbb{N} \backslash \{1\}$ such that $x(0^{11}_{1k}) = 0$ and $x(0^{11}_{1\ell}) = 1$. Let $j \in \mathbb{N} \backslash \{1, \ell\}$. We shall prove that $x(0^{11}_{1j}) = 0$. We have

$$\begin{array}{ccccccccc}
0^{11}_{1\ell} & & & & 0^{11}_{1k} & & & & 0^{11}_{1j} \\
\downarrow{\scriptstyle u_1} & & & & \downarrow{\scriptstyle u_1} & & & & \downarrow{\scriptstyle u_1} \\
* & & & & * & & & & * \\
\uparrow{\scriptstyle u_1} & & & & \uparrow{\scriptstyle u_1} & & & & \uparrow{\scriptstyle u_1} \\
0^{112}_{1\ell k} & \xrightarrow{u_0} & * & \xleftarrow{u_0} & 0^{112}_{1k\ell} & \xrightarrow{u_2} & * & \xleftarrow{u_2} & 0^{112}_{1j\ell}
\end{array}$$

in $\mathbf{A}$. Since $x(0^{112}_{1k\ell}) \in \{0,1,2\}$, applying the homomorphism $x$ gives us

$$\begin{array}{ccccccccc}
\boxed{1} & & & & \boxed{0} & & & & x(0^{11}_{1j}) \\
\downarrow{\scriptstyle u_1} & & & & \downarrow{\scriptstyle u_1} & & & & \downarrow{\scriptstyle u_1} \\
* & & & & * & & & & * \\
\uparrow{\scriptstyle u_1} & & & & \uparrow{\scriptstyle u_1} & & & & \uparrow{\scriptstyle u_1} \\
1 & \xrightarrow{u_0} & * & \xleftarrow{u_0} & 2 & \xrightarrow{u_2} & * & \xleftarrow{u_2} & 2
\end{array}$$

in $\underline{\mathbf{M}}$. As $x(0^{11}_{1j}) \in \{0,1\}$, it follows that $x(0^{11}_{1j}) = 0$. So $A_0 \backslash \{0^{11}_{1\ell}\}$ is the unique non-trivial block of $\ker(x{\restriction}_{A_0})$.

Define $g \in M^{\mathbb{N}}$ by $g(n) := \rho_n(a_n)$, where $a_n$ is any element of the non-trivial block of $\ker(\rho_n{\restriction}_{A_0})$. Then $g = 0^1_1$. But $0^1_1 \notin \mathrm{sg}_{\underline{\mathbf{M}}^{\mathbb{N}}}(A_0 \cup B)$, as $\underline{\mathbf{M}}$ is a unary algebra. So $g \notin A$ and therefore $\underline{\mathbf{M}}$ is not dualisable, by Lemma 5.2. □

**Corollary 5.6.** *No three-kernel three-element unary algebra is dualisable.*

We have now proved the characterisation of dualisable three-element unary algebras given in the introduction. Claim (i) of the theorem follows from Theorems 2.7 and 3.9. Claim (ii) of the theorem follows from Theorems 4.3, 4.4, 4.10, 5.3 and 5.4. Claim (iii) holds by Corollary 5.6.

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