



ELSEVIER

Journal of Pure and Applied Algebra 169 (2002) 1–28

JOURNAL OF
PURE AND
APPLIED ALGEBRA

www.elsevier.com/locate/jpaa

Binary homomorphisms and natural dualities

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Received 10 October 1999; received in revised form 5 March 2001

Communicated by J. Adámek

Abstract

We investigate ways in which certain binary homomorphisms of a finite algebra can guarantee its dualisability. Of particular interest are those binary homomorphisms which are lattice, flat-semilattice or group operations. We prove that a finite algebra which has a pair of lattice operations amongst its binary homomorphisms is dualisable. As an application of this result, we find that every finite unary algebra can be embedded into a dualisable algebra. We develop some general tools which we use to prove the dualisability of a large number of unary algebras. For example, we show that the endomorphisms of a finite cyclic group are the operations of a dualisable unary algebra. © 2002 Elsevier Science B.V. All rights reserved.

MSC: Primary 08C15; secondary 18A40; 06D50

One of the most fundamental questions in the theory of natural dualities is: “*Which finite algebras are dualisable?*” More specifically, we seek an algorithm to decide, given a finite algebra, whether or not it is dualisable. Experience continues to suggest that the general dualisability problem is very difficult, and that there may well be no such algorithm. This situation leads us to restrict the dualisability problem to special classes of algebras. For example, it has been completely solved for commutative rings with identity [3]:

- *a finite commutative ring with identity is dualisable if and only if its Jacobson radical is self annihilating.*

Within congruence-distributive varieties, the dualisability problem has been reduced to the problem of deciding if the algebra has a near-unanimity term ([8,4]-if; [6]-only if):

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- a finite algebra that generates a congruence-distributive variety is dualisable if and only if it has a near-unanimity term.

Recently, Davey and Willard [9] and Saramago [15] proved that a quasi-variety which is dualisable with respect to one finite generator is also dualisable with respect to every other finite generator. Applying this fact to the examples given in [1], we find many quasi-varieties in which every finite member is dualisable. For example, every finite abelian group, semilattice or vector space is dualisable. Partial information is given about the dualisability problem for groups in [7,13,14]. Beyond these results, knowledge of dualisability remains piecemeal and scattered.

In order to see just how difficult the dualisability problem can be, we examine its restriction to what might be considered the simplest possible algebras: those with only unary operations. The richness of the dualisability problem for unary algebras is strikingly illustrated by a discovery of Hyndman and Willard [11]. They answered a longstanding question by exhibiting an algebra which is dualisable but not fully dualisable. Their example has only three elements and two unary operations! Our goal is not to find dualities that are likely to be useful, since unary algebras are already very simple. Rather, by attempting to solve the dualisability problem for unary algebras, we hope to shed new light on the general problem. In this paper we present a general technique for proving the dualisability of a finite algebra $\underline{\mathbf{M}}$ which arose from our study of unary algebras. The basic idea is to give sufficient conditions for the dualisability of $\underline{\mathbf{M}}$ based on the existence of well-behaved binary homomorphisms $g: \underline{\mathbf{M}}^2 \rightarrow \underline{\mathbf{M}}$.

After outlining the relevant background in Section 1, we begin Section 2 with a quite specific example of what can be achieved. We show that if a finite algebra $\underline{\mathbf{M}}$ has binary homomorphisms \vee and \wedge such that $\langle M; \vee, \wedge \rangle$ is a lattice, then $\underline{\mathbf{M}}$ is dualisable. In particular, for all finite lattices \mathbf{M}_0 , every set F of endomorphisms of \mathbf{M}_0 determines a dualisable unary algebra $\underline{\mathbf{M}} := \langle M; F \rangle$. In Section 3, we investigate finite algebras that have elements s for which there is a binary homomorphism g satisfying $g(x, y) = s \Leftrightarrow x = y = s$. We show that the algebra is dualisable whenever every element is of this sort, and that otherwise these elements can be used to help establish dualisability. In Section 4 we study algebras which have a unary term function that is a retraction onto a dualisable subalgebra, and exhibit conditions under which this duality lifts to the entire algebra. Section 5 concerns algebras which have a flat-semilattice operation among their binary homomorphisms, and shows how this operation can be used to establish dualisability. In the last section we see how the existence of a binary homomorphism which is a group operation can also help lead to a duality.

Each of Sections 2, 3, 4, 5 and 6 ends with applications that produce specific new dualities. For example, Section 2 contains a proof that every finite unary algebra can be embedded into a dualisable algebra. This provides the first examples of dualisable algebras with non-dualisable subalgebras and thereby solves the Inherent Non-dualisability Problem (see 10.5.6 in [1]). One immediate consequence of our results in Section 3 is that if there are semilattice operations, $\wedge_1, \dots, \wedge_k$, which are binary homomorphisms

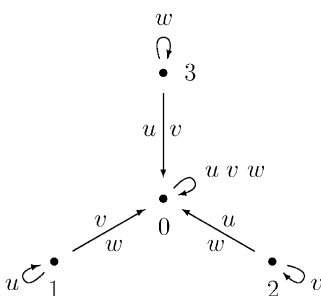


Fig. 1. $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3\}; u, v, w \rangle$.

of $\underline{\mathbf{M}}$, such that every element of M is a maximal element of at least one of the induced order relations, \leq_1, \dots, \leq_k , on M , then $\underline{\mathbf{M}}$ is dualisable. Another immediate consequence is that the endomorphisms of a prime cyclic group are the operations of a dualisable unary algebra. Using the tools established in Section 4, we extend this result in Section 6 to all finite cyclic groups. The authors expended a great deal of energy trying to prove the dualisability of the four-element unary algebra $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3\}; u, v, w \rangle$, where u, v and w act on $\{0, 1, 2, 3\}$ as illustrated in Fig. 1. The simplicity and symmetry of $\underline{\mathbf{M}}$ led the authors to believe that this algebra was dualisable. As we shall see in Theorem 37, it takes a large slab of our general theory plus some additional special tricks to prove that this intuition was correct.

1. Preliminaries

In this section we briefly review the notions and theorems that we require. All missing details and motivations may be found in [1]. Let $\underline{\mathbf{M}}$ be a finite algebra. An *algebraic relation on $\underline{\mathbf{M}}$* is a relation $r \subseteq M^n$, for some $n \in \mathbb{N}$, such that r forms a subalgebra of $\underline{\mathbf{M}}^n$. An *algebraic operation on $\underline{\mathbf{M}}$* is an operation $g: M^n \rightarrow M$, for some $n \in \mathbb{N} \cup \{0\}$, such that g is a homomorphism from $\underline{\mathbf{M}}^n$ to $\underline{\mathbf{M}}$. An *algebraic partial operation on $\underline{\mathbf{M}}$* is a map $h: \text{dom}(h) \rightarrow M$, with $\text{dom}(h) \subseteq M^n$ for some $n \in \mathbb{N}$, such that $\text{dom}(h)$ forms a subalgebra of $\underline{\mathbf{M}}^n$ and h is a homomorphism from this algebra to $\underline{\mathbf{M}}$. We say that algebraic relations on $\underline{\mathbf{M}}$, algebraic operations on $\underline{\mathbf{M}}$ and algebraic partial operations on $\underline{\mathbf{M}}$ are all *algebraic over $\underline{\mathbf{M}}$* . A structure $\underline{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ is called an *alter ego of $\underline{\mathbf{M}}$* if

- (i) G is a set of algebraic operations on $\underline{\mathbf{M}}$,
- (ii) H is a set of algebraic partial operations on $\underline{\mathbf{M}}$,
- (iii) R is a set of algebraic relations on $\underline{\mathbf{M}}$, and
- (iv) \mathcal{T} is the discrete topology on M .

We want to use the alter ego $\underline{\mathbf{M}}$ to try to set up a natural representation for the quasi-variety $\mathbb{I}\text{SP}\underline{\mathbf{M}}$.

For each algebra \mathbf{A} in the category $\mathcal{A} := \mathbb{I}\mathbb{S}\mathbb{P}\mathbf{M}$, we define the *dual* $D(\mathbf{A})$ to be the homset $\mathcal{A}(\mathbf{A}, \mathbf{M})$ regarded as a substructure of \mathbf{M}^A . Thus $D(\mathbf{A})$ belongs to the category $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+\mathbf{M}$ of all isomorphic copies of topologically closed substructures of non-zero powers of \mathbf{M} . The *dual* $E(\mathbf{X})$ of a structure $\mathbf{X} \in \mathcal{X}$ is the homset $\mathcal{X}(\mathbf{X}, \mathbf{M})$ regarded as a subalgebra of \mathbf{M}^X . There is a natural evaluation map $e_{\mathbf{A}} : \mathbf{A} \rightarrow ED(\mathbf{A})$, given by $e_{\mathbf{A}}(a)(x) := x(a)$ for all $a \in A$ and each $x \in D(\mathbf{A})$. The map $e_{\mathbf{A}}$ is an embedding, since \mathbf{A} belongs to the quasi-variety $\mathbb{I}\mathbb{S}\mathbb{P}\mathbf{M}$. For each $a \in A$, we say that the map $e_{\mathbf{A}}(a)$ is *given by evaluation at a* and that $e_{\mathbf{A}}(a)$ is an *evaluation*. For each $Y \subseteq D(\mathbf{A})$, a map $\alpha : D(\mathbf{A}) \rightarrow M$ is an *evaluation on Y* if $\alpha|_Y = e_{\mathbf{A}}(a)|_Y$ for some $a \in A$.

We say that \mathbf{M} *yields a duality on \mathbf{A}* if $e_{\mathbf{A}}$ is surjective. In this case, the map $e_{\mathbf{A}}$ is an isomorphism from \mathbf{A} onto the algebra of all continuous homomorphisms from its dual, $D(\mathbf{A})$, into \mathbf{M} . Thus, \mathbf{M} yields a duality on \mathbf{A} if every morphism $\alpha : D(\mathbf{A}) \rightarrow \mathbf{M}$ is an evaluation. The structure \mathbf{M} *yields a duality on \mathcal{A}* if $e_{\mathbf{A}}$ is surjective for all $\mathbf{A} \in \mathcal{A}$. If \mathbf{M} yields a duality on \mathcal{A} , then we also say that \mathbf{M} is a *dualising structure* for \mathbf{M} or that \mathbf{M} *dualises \mathbf{M}* . Finally, we say that \mathbf{M} is *dualisable* if a dualising structure for \mathbf{M} exists.

Now let \mathbf{A} be an algebra, not necessarily in $\mathbb{I}\mathbb{S}\mathbb{P}\mathbf{M}$, but of the same type as \mathbf{M} . It is easy to check that the set of all homomorphisms from \mathbf{A} to \mathbf{M} can still be viewed as a closed substructure $D(\mathbf{A})$ of \mathbf{M}^A . Moreover, it is relatively straightforward to prove that $e_{\mathbf{A}} : \mathbf{A} \rightarrow ED(\mathbf{A})$ is surjective for all $\mathbf{A} \in \mathbb{I}\mathbb{S}\mathbb{P}\mathbf{M}$ if and only if $e_{\mathbf{A}}$ is surjective for all algebras \mathbf{A} of the same type as \mathbf{M} . (Show that, for any algebra \mathbf{A} with same type as \mathbf{M} , the structure $D(\mathbf{A})$ is isomorphic to $D(\mathbf{A}/\Theta)$, where $\Theta := \bigwedge \{\theta \in \text{Con}(\mathbf{A}) \mid \mathbf{A}/\theta \in \mathbb{I}\mathbb{S}\mathbb{P}\mathbf{M}\}$, and then do a diagram chase.) Thus, unless stated otherwise, we will assume that \mathbf{A} is an arbitrary algebra of the same type as \mathbf{M} .

For each $n \in \mathbb{N}$, let R_n denote the set of all n -ary algebraic relations on \mathbf{M} , and let G_n denote the set of all n -ary algebraic operations on \mathbf{M} . In particular, the set G_2 of *binary homomorphisms* of \mathbf{M} will play a central role in this study.

Lemma 1. *Let \mathbf{M} be a finite algebra, and let $\alpha : D(\mathbf{A}) \rightarrow M$ and let $n \in \mathbb{N}$. Then α preserves (the relations in) R_n if and only if α agrees with an evaluation on each subset of $D(\mathbf{A})$ with at most n elements.*

Proof. Assume that α preserves R_n and let $x_1, \dots, x_n \in D(\mathbf{A})$. The relation $r := \{(x_1(a), \dots, x_n(a)) \mid a \in A\}$ belongs to R_n , so r is preserved by α . We have $(x_1, \dots, x_n) \in r^{D(\mathbf{A})}$ and therefore $(\alpha(x_1), \dots, \alpha(x_n)) \in r$. Thus there is an $a \in A$ such that $\alpha(x_i) = x_i(a) = e_{\mathbf{A}}(a)(x_i)$, for all $i \in \{1, \dots, n\}$. The other direction is easy. \square

We require two theorems which give sufficient conditions for \mathbf{M} to be dualisable. If $\mathbf{M} = \langle M; G, R, \mathcal{F} \rangle$, that is, if the set H of partial operations is empty, then we refer to \mathbf{M} as a *total structure*. The structure \mathbf{M} is said to be of *finite type* provided $G \cup H \cup R$ is finite.

Second Duality Theorem 2 (Davey and Werner [8]; see [1,2.2.7]). *Let $\underline{\mathbf{M}}$ be a finite algebra and let $\tilde{\mathbf{M}}$ be a total structure, with R finite, which is algebraic over $\underline{\mathbf{M}}$. Then $\tilde{\mathbf{M}}$ yields a duality on $\mathcal{A} := \mathbb{I}\text{SP}\underline{\mathbf{M}}$ and $\tilde{\mathbf{M}}$ is injective in the category $\mathcal{X} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+\tilde{\mathbf{M}}$ provided the following interpolation condition holds:*

(IC) *for every $n \in \mathbb{N}$ and every substructure \mathbf{X} of $\tilde{\mathbf{M}}^n$, each morphism $\alpha : \mathbf{X} \rightarrow \tilde{\mathbf{M}}$ extends to an n -ary term function of $\tilde{\mathbf{M}}$.*

Duality Compactness Theorem 3 (Willard [3], Zadori [16]; see [1,2.2.11]). *Let $\underline{\mathbf{M}}$ be a finite algebra and let $\tilde{\mathbf{M}}$ be an alter ego of $\underline{\mathbf{M}}$ with finite type. Then $\tilde{\mathbf{M}}$ yields a duality on $\mathcal{A} := \mathbb{I}\text{SP}\underline{\mathbf{M}}$ provided it yields a duality on each finite algebra in \mathcal{A} .*

The Second Duality Theorem has the advantage that it gives us a duality with the added bonus of the injectivity of $\tilde{\mathbf{M}}$ in \mathcal{X} , but it requires that we consider more morphisms into $\tilde{\mathbf{M}}$. The Duality Compactness Theorem, on the other hand, has the advantage that it allows the use of partial operations in the type of $\tilde{\mathbf{M}}$, but this type must be finite.

Assume that the algebra $\underline{\mathbf{M}} = \langle M; F \rangle$ has, amongst its binary homomorphisms, maps g_1, \dots, g_n such that the algebra $\mathbf{M}_0 := \langle M; g_1, \dots, g_n \rangle$ is a well-behaved, familiar algebra. Then we may regard \mathbf{M}_0 as the zeroth approximation to a dualising structure for $\underline{\mathbf{M}}$. Our strategy will be to use our knowledge of the quasi-variety generated by \mathbf{M}_0 to find a finite family r_1, \dots, r_k of algebraic relations on $\underline{\mathbf{M}}$ such that

$$\tilde{\mathbf{M}} := \langle M; g_1, \dots, g_n, r_1, \dots, r_k, \mathcal{F} \rangle$$

is a dualising structure for $\underline{\mathbf{M}}$.

2. Lattice operations

To illustrate our strategy, we give a beautiful example which uses a pair of binary homomorphisms which are lattice operations to prove the dualisability of a finite algebra.

Theorem 4. *Let $\underline{\mathbf{M}}$ be a finite algebra which has binary homomorphisms \vee and \wedge such that $\langle M; \vee, \wedge \rangle$ is a lattice. Then $\tilde{\mathbf{M}} := \langle M; \vee, \wedge, R_{2|M|}, \mathcal{F} \rangle$ yields a duality on $\mathbb{I}\text{SP}\underline{\mathbf{M}}$.*

Proof. We shall use the Duality Compactness Theorem. Let \mathbf{A} be finite and let $\alpha : \mathbf{D}(\mathbf{A}) \rightarrow \tilde{\mathbf{M}}$ be a morphism. We want to show that α is an evaluation. First set $\alpha(\mathbf{D}(\mathbf{A})) = \{a_1, \dots, a_k\} \subseteq M$. For each $\ell \in \{1, \dots, k\}$, define $x_\ell := \bigwedge \alpha^{-1}(a_\ell)$ and $y_\ell := \bigvee \alpha^{-1}(a_\ell)$. By Lemma 1, we know that there is an $a \in A$ such that α agrees with $e_{\mathbf{A}}(a)$ on the set $\{x_1, y_1, \dots, x_k, y_k\}$. Now consider any $z \in \mathbf{D}(\mathbf{A})$. There is some

$\ell \in \{1, \dots, k\}$ such that $\alpha(z) = a_\ell$. We have $x_\ell \leq z \leq y_\ell$. Since α preserves \vee and \wedge , we also have $\alpha(x_\ell) = a_\ell = \alpha(y_\ell)$. So

$$a_\ell = \alpha(x_\ell) = x_\ell(a) \leq z(a) \leq y_\ell(a) = \alpha(y_\ell) = a_\ell,$$

and consequently $\alpha(z) = a_\ell = z(a) = e_A(a)(z)$. Thus $\underline{\mathbf{M}}$ yields a duality on $\mathbb{ISP}\underline{\mathbf{M}}$, by the Duality Compactness Theorem. \square

The next result follows immediately from Theorem 4. Note that a lattice $\mathbf{M}_0 = \langle M; \vee, \wedge \rangle$ is algebraic over a unary algebra $\underline{\mathbf{M}} = \langle M; F \rangle$ if and only if F is a set of endomorphisms of the lattice \mathbf{M}_0 .

Theorem 5. *Lattice endomorphisms yield dualisable unary algebras. More precisely, if $\underline{\mathbf{M}} = \langle M; F \rangle$ is a finite unary algebra such that $F \subseteq \text{End}(\mathbf{M}_0)$, for some lattice $\mathbf{M}_0 = \langle M; \vee, \wedge \rangle$, then $\underline{\mathbf{M}}$ is dualised by $\underline{\mathbf{M}} := \langle M; \vee, \wedge, R_{2|M|}, \mathcal{T} \rangle$.*

Corollary 6. *Let $\underline{\mathbf{M}} = \langle M; F \rangle$ be a finite unary algebra. If there is a total order on M which is preserved by each $u \in F$, then $\underline{\mathbf{M}}$ is dualisable.*

The previous result is quite powerful when applied to three-element unary algebras. The authors used a C program to analyse the essentially unary clones on the set $\{0, 1, 2\}$. There are exactly 699 essentially unary clones on $\{0, 1, 2\}$. Of them, precisely 221 consist of order-preserving maps for some total order on $\{0, 1, 2\}$ and therefore determine dualisable unary algebras. The dualisable three-element unary algebras are completely characterised in the paper [2].

A finite algebra $\underline{\mathbf{M}}$ is said to be *inherently non-dualisable* if $\underline{\mathbf{M}} \in \mathbb{ISP}\underline{\mathbf{N}}$ implies that $\underline{\mathbf{N}}$ is non-dualisable, for all finite algebras $\underline{\mathbf{N}}$. The Inherent Non-dualisability Problem (10.5.6 [1]) asks whether there is a finite algebra which is non-dualisable but not inherently non-dualisable, and, in particular, whether there is a dualisable algebra which has a non-dualisable subalgebra. A simple application of Theorem 5 tells us that the answer in both cases is ‘yes’. There is an abundance of non-dualisable unary algebras. For example, if M has at least three elements and $\{0, 1\}$ is a subset of M , then both $\langle M; \{0, 1\}^M \rangle$ and $\langle M; M^M \rangle$ are non-dualisable (see [10] and Chapter 10 of [1]).

Theorem 7. *There is no inherently non-dualisable unary algebra. Indeed, every finite unary algebra can be embedded into a dualisable algebra.*

Proof. Let $\underline{\mathbf{M}} = \langle M; F \rangle$ be a finite unary algebra and let $\mathbf{N}_0 = \langle N; \vee, \wedge \rangle$ be the free distributive lattice generated by the set M . Every operation $u \in F$ has a (unique) extension to a lattice endomorphism (also denoted by u) of \mathbf{N}_0 . Thus $\underline{\mathbf{M}}$ is a subalgebra of $\underline{\mathbf{N}} := \langle N; F \rangle$ and $\underline{\mathbf{N}}$ is dualisable by Theorem 5. \square

The driving force behind the proof of Theorem 4 is the fact that every congruence class z/θ of the lattice congruence $\theta := \ker(\alpha)$ on $D(\mathbf{A})$ is an interval $[x, y]$ and hence is the principal congruence class $x/\theta_{D(\mathbf{A})}(x, y)$. In fact, it suffices to know that every congruence class of θ is the union of some fixed finite number of principal congruence classes. Let $n \in \mathbb{N}$. An algebra \mathbf{A} satisfies the *n-Principal Congruence Class Property* if every congruence class z/θ of a congruence θ on \mathbf{A} is the union of at most n principal congruence classes of the form $x/\theta_{\mathbf{A}}(x, y)$, with $x, y \in z/\theta$. Thus every finite lattice satisfies the 1-Principal Congruence Class Property.

Assume that $\mathbf{M}_0 = \langle M; G \rangle$ is a finite algebra which is algebraic over $\underline{\mathbf{M}}$ and assume that every finite algebra in $\mathbb{ISP}\mathbf{M}_0$ satisfies the *n-Principal Congruence Class Property*. We could alter the proof of Theorem 4 to show that $\underline{\mathbf{M}}$ is dualisable. Instead, we shall restrict our focus to the case that $\underline{\mathbf{M}}$ is a unary algebra and obtain a stronger conclusion.

Theorem 8. *Let $n \in \mathbb{N}$, let $\mathbf{M}_0 = \langle M; G \rangle$ be a finite algebra and assume that every finite algebra in $\mathbb{ISP}\mathbf{M}_0$ satisfies the *n-Principal Congruence Class Property*. Then every subset F of $\text{End}(\mathbf{M}_0)$ yields a dualisable unary algebra $\underline{\mathbf{M}} := \langle M; F \rangle$. Indeed, $\tilde{\mathbf{M}} := \langle M; G, R_{2n|M|}, \mathcal{F} \rangle$ yields a duality on $\mathcal{A} := \mathbb{ISP}\tilde{\mathbf{M}}$ such that $\tilde{\mathbf{M}}$ is injective in the dual category $\mathcal{X} := \mathbb{IS}_c\mathbb{P}^+\tilde{\mathbf{M}}$.*

Proof. Assume that F forms a monoid under composition. We will use the Second Duality Theorem. Let \mathbf{X} be a substructure of $\tilde{\mathbf{M}}^m$, for some $m \in \mathbb{N}$, and let $\alpha : \mathbf{X} \rightarrow \underline{\mathbf{M}}$ be a morphism. Let \mathbf{X}_0 denote the subalgebra of $(\mathbf{M}_0)^m$ determined by X . Since $\alpha : \mathbf{X}_0 \rightarrow \underline{\mathbf{M}}$ is a homomorphism, $\theta := \ker(\alpha)$ is a congruence on \mathbf{X}_0 and $\mathbf{N}_0 := \alpha(\mathbf{X}_0)$ is a subalgebra of $\underline{\mathbf{M}}$.

Let $N = \{a_1, \dots, a_k\}$ and define the relation $r \subseteq M^{2kn}$ by

$$r := M^{2kn} \setminus \{b_{11}b'_{11} \dots b_{1n}b'_{1n} \dots b_{k1}b'_{k1} \dots b_{kn}b'_{kn} \mid (\exists u \in F)(\forall \ell, j) u(b_{\ell j}) = u(b'_{\ell j}) = a_{\ell}\}.$$

It is easy to see that r is algebraic over $\underline{\mathbf{M}}$. Since $2kn \leq 2n|M|$, the morphism α preserves r . As the algebra \mathbf{X}_0 satisfies the *n-Principal Congruence Class Property*, for all ℓ there exist $x_{\ell 1}, y_{\ell 1}, \dots, x_{\ell n}, y_{\ell n} \in X$ such that

$$\alpha^{-1}(a_{\ell}) = x_{\ell 1}/\theta_{\mathbf{X}_0}(x_{\ell 1}, y_{\ell 1}) \cup \dots \cup x_{\ell n}/\theta_{\mathbf{X}_0}(x_{\ell n}, y_{\ell n}).$$

As

$$\alpha(x_{11})\alpha(y_{11}) \dots \alpha(x_{1n})\alpha(y_{1n}) \dots \alpha(x_{k1})\alpha(y_{k1}) \dots \alpha(x_{kn})\alpha(y_{kn})$$

belongs to $M^{2kn} \setminus r$ and α preserves r , there exists $i \in \{1, \dots, m\}$ and there exists $u \in F$ such that, for all ℓ and j , we have $u(x_{\ell j}(i)) = u(y_{\ell j}(i)) = a_{\ell}$, that is, $(u \circ \pi_i)(x_{\ell j}) = (u \circ \pi_i)(y_{\ell j}) = a_{\ell}$. It follows that

$$\theta_{\mathbf{X}_0}(x_{\ell j}, y_{\ell j}) \leq \ker(u \circ \pi_i) \quad \text{and} \quad (u \circ \pi_i)(x_{\ell j}) = a_{\ell} \tag{*}$$

for all ℓ and j . Let $z \in X$. Then $\alpha(z) = a_\ell$, for some ℓ , and we have

$$\begin{aligned} z \in \alpha^{-1}(a_\ell) &= x_{\ell 1}/\theta_{\mathbf{X}_0}(x_{\ell 1}, y_{\ell 1}) \cup \cdots \cup x_{\ell n}/\theta_{\mathbf{X}_0}(x_{\ell n}, y_{\ell n}) \\ &\Rightarrow (\exists j) z \in x_{\ell j}/\theta_{\mathbf{X}_0}(x_{\ell j}, y_{\ell j}) \\ &\Rightarrow (u \circ \pi_i)(z) = (u \circ \pi_i)(x_{\ell j}) = a_\ell = \alpha(z), \text{ by } (*). \end{aligned}$$

Thus $\alpha = (u \circ \pi_i) \upharpoonright_X$. Hence (IC) holds and so, by the Second Duality Theorem, the structure $\underline{\mathbf{M}}$ yields a duality on \mathcal{A} and $\underline{\mathbf{M}}$ is injective in \mathcal{X} . \square

Corollary 9. *If $\underline{\mathbf{M}} = \langle M; F \rangle$ is a finite unary algebra with $F \subseteq \text{End}(\mathbf{M}_0)$, for some lattice $\mathbf{M}_0 = \langle M; \vee, \wedge \rangle$, then $\underline{\mathbf{M}} := \langle M; \vee, \wedge, R_{2|M|}, \mathcal{T} \rangle$ yields a duality on $\mathcal{A} := \|\text{SP}\underline{\mathbf{M}}\|$ and is injective in $\mathcal{X} := \|\mathbb{S}_c\mathbb{P}^+\underline{\mathbf{M}}\|$.*

3. Operations with strong idempotents

We begin this section by considering what it means for a finite algebra $\underline{\mathbf{M}}$ to be dualised by an alter ego $\underline{\mathbf{M}}$. Let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ be a morphism and let $a \in A$. Then $\alpha = e_{\mathbf{A}}(a)$ if and only if $a \in x^{-1}(\alpha(x))$ for each $x \in D(\mathbf{A})$. It follows that α is an evaluation if and only if the set

$$A_\alpha := \bigcap \{x^{-1}(\alpha(x)) \mid x \in D(\mathbf{A})\}$$

is not empty, in which case $\alpha = e_{\mathbf{A}}(a)$ for each $a \in A_\alpha$.

We will present a method for identifying a set $S \subseteq M$ and a structure $\underline{\mathbf{M}}$ such that, for every morphism $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$, the set

$$A_{\alpha, S} := \bigcap \{x^{-1}(\alpha(x)) \mid x \in D(\mathbf{A}) \text{ and } \alpha(x) \in S\}$$

is not empty, that is, α is an evaluation on $\alpha^{-1}(S)$. The first choice of S will be a set $\Sigma_{\underline{\mathbf{M}}}$ of elements of M which, in a natural sense, are strong idempotents of some binary homomorphisms of $\underline{\mathbf{M}}$. We will then develop a general technique, in the form of the GST Lemmas below, to extend the structure on $\underline{\mathbf{M}}$ in a way that will guarantee that α is an evaluation on all of $D(\mathbf{A})$. This technique will depend on $\underline{\mathbf{M}}$ having the right binary homomorphisms.

Lemma 1 gives a simple condition which guarantees that a map $\alpha: D(\mathbf{A}) \rightarrow M$ is pointwise an evaluation.

Lemma 10. *Let $\underline{\mathbf{M}}$ be a finite algebra and assume that $\alpha: D(\mathbf{A}) \rightarrow M$ preserves R_1 . Then, for each $x \in D(\mathbf{A})$, the set $x^{-1}(\alpha(x))$ is not empty and $\alpha(x) = x(a)$ for all $a \in x^{-1}(\alpha(x))$.*

For a map $\alpha: D(\mathbf{A}) \rightarrow M$, we next fix an element $s \in M$ and ask when the set

$$A_{\alpha, s} := \bigcap \{x^{-1}(\alpha(x)) \mid x \in D(\mathbf{A}) \text{ and } \alpha(x) = s\}$$

is non-empty. If \mathbf{A} is finite, a natural condition to ensure $A_{\alpha, s} \neq \emptyset$ is that $x^{-1}(\alpha(x)) \neq \emptyset$, for each $x \in \alpha^{-1}(s)$, and that the set

$$\mathcal{X}_{\alpha, s} := \{x^{-1}(\alpha(x)) \mid x \in D(\mathbf{A}) \text{ and } \alpha(x) = s\}$$

is closed under intersection. This turns out to be equivalent to a condition on binary homomorphisms.

Lemma 11. *Let $\underline{\mathbf{M}}$ be a finite algebra and let $s \in M$. Then the following are equivalent:*

(i) *there is an alter ego $\underline{\mathbf{M}}$ of $\underline{\mathbf{M}}$ such that $\mathcal{X}_{\alpha, s}$ is closed under intersection for all \mathbf{A} and all morphisms $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$;*

(ii) *there is a binary homomorphism $g_s \in G_2$ such that $g_s^{-1}(s) = \{(s, s)\}$.*

Moreover, if $g_s \in G_2$ and $g_s^{-1}(s) = \{(s, s)\}$, then $\mathcal{X}_{\alpha, s}$ is closed under intersection for all \mathbf{A} and all $\alpha: D(\mathbf{A}) \rightarrow M$ which preserve g_s .

Proof. Assume (i) and define $\mathbf{A} := \underline{\mathbf{M}}^2$ and $\alpha := e_{\mathbf{A}}(s, s)$. The projections π_1 and π_2 belong to $D(\mathbf{A})$. For each $i \in \{1, 2\}$, we have $\alpha(\pi_i) = s$ and so $\pi_i^{-1}(s) \in \mathcal{X}_{\alpha, s}$. By (i), there is some $g_s \in D(\mathbf{A})$ such that $g_s^{-1}(s) = \pi_1^{-1}(s) \cap \pi_2^{-1}(s) = \{(s, s)\}$.

Now assume (ii) and define $\underline{\mathbf{M}} := \langle M; g_s, \mathcal{F} \rangle$ to prove both (i) and ‘Moreover’. Let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ and let $x, y \in \alpha^{-1}(s)$. Then $g_s(x, y) \in D(\mathbf{A})$ with

$$\alpha(g_s(x, y)) = g_s(\alpha(x), \alpha(y)) = g_s(s, s) = s$$

and $x^{-1}(s) \cap y^{-1}(s) = g_s(x, y)^{-1}(s) \in \mathcal{X}_{\alpha, s}$. \square

In view of Lemma 11, we say that $s \in M$ is a *strong idempotent* of the binary homomorphism g of $\underline{\mathbf{M}}$, and that g is a *strong idempotent function* for s , if $g^{-1}(s) = \{(s, s)\}$. We denote the set of all strong idempotents of $\underline{\mathbf{M}}$ by

$$\Sigma_{\underline{\mathbf{M}}} := \{s \in M \mid (\exists g_s \in G_2) g_s^{-1}(s) = \{(s, s)\}\}.$$

In- Σ Lemma 12. *Let $\underline{\mathbf{M}}$ be a finite algebra, let $S \subseteq \Sigma_{\underline{\mathbf{M}}}$ and let \mathbf{A} be finite. Assume that $\alpha: D(\mathbf{A}) \rightarrow M$ preserves a strong idempotent function for each $s \in S$ and preserves each relation in $R_{|S|}$. Then α is an evaluation on $\alpha^{-1}(S)$.*

Proof. Let $S \cap \alpha(D(\mathbf{A})) = \{s_1, \dots, s_k\}$. Since \mathbf{A} is finite, we can use Lemma 11 to find, for each $i \in \{1, \dots, k\}$, an $x_i \in D(\mathbf{A})$ such that $\alpha(x_i) = s_i$ and $x_i^{-1}(s_i) = A_{\alpha, s_i}$. By Lemma 1, there is an $a \in A$ such that $\alpha(x_i) = x_i(a)$, for all $i \in \{1, \dots, k\}$. We have $a \in \bigcap_{i=1}^k A_{\alpha, s_i} = A_{\alpha, S}$. So α is an evaluation on $\alpha^{-1}(S)$. \square

Applying the In- Σ Lemma, we are now able to prove that some algebras are dualisable just by looking at their binary homomorphisms.

In- Σ Theorem 13. Let $\underline{\mathbf{M}}$ be a finite algebra such that every element $s \in M$ is a strong idempotent of some $g_s \in G_2$, that is, $\Sigma_{\underline{\mathbf{M}}} = M$. Then $\underline{\mathbf{M}}$ is dualised by $\underline{\mathbf{M}} := \langle M; G, R_{|M|}, \mathcal{T} \rangle$, where $G := \{g_s \mid s \in M\}$.

If g is a meet-semilattice operation on M , then $g^{-1}(s) = \{(s, s)\}$ if and only if s is maximal with respect to the order induced on M by g . The following corollary is a trivial though useful consequence of the In- Σ Theorem.

Corollary 14. Let $\underline{\mathbf{M}}$ be a finite algebra such that there is a set G of algebraic meet-semilattice operations on $\underline{\mathbf{M}}$. Then $\Sigma_{\underline{\mathbf{M}}}$ contains the set S of all elements of M which are maximal with respect to the order induced by at least one of the operations in G . If $S = M$, then $\underline{\mathbf{M}} := \langle M; G, R_{|M|}, \mathcal{T} \rangle$ dualises $\underline{\mathbf{M}}$.

Let \mathbf{A} be finite, let $\alpha: D(\mathbf{A}) \rightarrow M$ and define

$$\mathcal{X}_\alpha := \{x^{-1}(\alpha(x)) \mid x \in D(\mathbf{A})\}.$$

We can order the elements of \mathcal{X}_α by set inclusion. If $\Sigma_{\underline{\mathbf{M}}} = M$ and α preserves all the strong idempotent functions of $\underline{\mathbf{M}}$, then \mathcal{X}_α has at most $|M|$ minimal elements: for each $s \in M$, the set $\mathcal{X}_{\alpha, s}$ is closed under intersection, and so the minimal elements of \mathcal{X}_α belong to $\{\bigcap \mathcal{X}_{\alpha, s} \mid s \in M\}$. In fact, any choice of $\underline{\mathbf{M}}$ which imposes a uniform finite upper bound on the number of minimal elements in the sets \mathcal{X}_α , for \mathbf{A} finite and $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$, will lead to a duality.

Theorem 15. Let $\underline{\mathbf{M}}$ be a finite algebra and let $\underline{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ be an alter ego of $\underline{\mathbf{M}}$ with finite type. Assume there is some $n \in \mathbb{N}$ such that, for all finite algebras $\mathbf{A} \in \mathbb{I}\mathbb{S}\mathbb{P}\underline{\mathbf{M}}$ and for every morphism $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$, the set \mathcal{X}_α has at most n minimal elements. Then $\underline{\mathbf{M}}' := \langle M; G, H, R \cup R_n, \mathcal{T} \rangle$ is a dualising structure for $\underline{\mathbf{M}}$.

Proof. Let $\mathbf{A} \in \mathbb{I}\mathbb{S}\mathbb{P}\underline{\mathbf{M}}$ be a finite algebra and let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}'$ be a morphism. Choose $x_1, \dots, x_n \in D(\mathbf{A})$ so that $\{x_i^{-1}(\alpha(x_i)) \mid i \in \{1, \dots, n\}\}$ includes all the minimal elements of \mathcal{X}_α . By Lemma 1, there is an $a \in A$ such that $\alpha(x_i) = x_i(a)$ for all $i \in \{1, \dots, n\}$. Now let $y \in D(\mathbf{A})$. For some $i \in \{1, \dots, n\}$, we have $a \in x_i^{-1}(\alpha(x_i)) \subseteq y^{-1}(\alpha(y))$, and so $\alpha(y) = y(a)$. The result follows by the Duality Compactness Theorem. \square

An interesting problem is that of finding an application of Theorem 15 which does not already follow from the In- Σ Theorem 13.

We now turn our attention to finite algebras $\underline{\mathbf{M}}$ for which $\Sigma_{\underline{\mathbf{M}}} \neq M$. Let \mathbf{A} be finite and assume that $\alpha: D(\mathbf{A}) \rightarrow M$ preserves all the strong idempotent functions of $\underline{\mathbf{M}}$ and preserves $R_{|\Sigma_{\underline{\mathbf{M}}}|}$. The In- Σ Lemma 12 tells us that α is an evaluation on $\alpha^{-1}(\Sigma_{\underline{\mathbf{M}}})$.

We will show that if α preserves some other special binary homomorphisms of $\underline{\mathbf{M}}$, then we can guarantee that α is an evaluation on all of $D(\mathbf{A})$.

Lemma 16. *Let $\underline{\mathbf{M}}$ be a finite algebra and assume that $\alpha: D(\mathbf{A}) \rightarrow M$ preserves the binary homomorphism g of $\underline{\mathbf{M}}$. Let $s, s' \in \alpha(D(\mathbf{A}))$ and assume that $a \in A_{\alpha, s} \cap A_{\alpha, s'}$. If $y \in D(\mathbf{A})$, then*

$$g(s, \alpha(y)) = s' \text{ implies } g(s, y(a)) = s'.$$

Proof. Let $y \in D(\mathbf{A})$ and assume that $g(s, \alpha(y)) = s'$. Choose $x \in D(\mathbf{A})$ such that $\alpha(x) = s$. Then

$$\alpha(g(x, y)) = g(\alpha(x), \alpha(y)) = g(s, \alpha(y)) = s'.$$

Since $a \in A_{\alpha, s} \cap A_{\alpha, s'}$, we have $x(a) = s$ and $g(x, y)(a) = s'$. So

$$s' = g(x, y)(a) = g(x(a), y(a)) = g(s, y(a)),$$

as required. \square

First GST Lemma 17. *Let $\underline{\mathbf{M}}$ be a finite algebra, let $S \subseteq M$ and let $t \in M \setminus S$. Let $\alpha: D(\mathbf{A}) \rightarrow M$ and assume there exist $n \in \mathbb{N}$, a set of binary homomorphisms $G = \{g_1, \dots, g_n\} \subseteq G_2$ and elements $s_1, s'_1, \dots, s_n, s'_n \in S \cap \alpha(D(\mathbf{A}))$ such that*

$$\bigg\& \bigg\limits_{i=1}^n g_i(s_i, m) = s'_i \Leftrightarrow m = t, \tag{\$}_t$$

for all $m \in M$. If α preserves G and is given by evaluation at $a \in A$ on $\alpha^{-1}(S)$, then α is also given by evaluation at a on $\alpha^{-1}(t)$.

Proof. Assume α agrees with $e_{\mathbf{A}}(a)$ on $\alpha^{-1}(S)$ and let $y \in D(\mathbf{A})$ such that $\alpha(y) = t$. By $(\$)_t$, we have $g_i(s_i, \alpha(y)) = s'_i$, for all $i \in \{1, \dots, n\}$. Using Lemma 16, we obtain $g_i(s_i, y(a)) = s'_i$, for each $i \in \{1, \dots, n\}$, and so, by $(\$)_t$, we conclude that $y(a) = t = \alpha(y)$. \square

We will give a number of applications of the First GST Lemma. In order to use this lemma to show that every α is an evaluation, it is in general necessary to check separately that each possible subset $\alpha(D(\mathbf{A})) \subseteq M$ contains the necessary elements s_i and s'_i . On occasion we can circumvent this difficulty by verifying a stronger hypothesis. For $t \in M$, we let $G_1(t)$ denote the set of images of t under the collection G_1 of endomorphisms of $\underline{\mathbf{M}}$.

Second GST Lemma 18. *Let $\underline{\mathbf{M}}$ be a finite algebra, let $S \subseteq M$ and let $t \in M \setminus S$. Assume there exist $n \in \mathbb{N}$, a set of binary homomorphisms $G = \{g_1, \dots, g_n\} \subseteq G_2$ and*

elements $s_1, s'_1, \dots, s_n, s'_n \in S \cap G_1(t)$ such that

$$\bigwedge_{i=1}^n g_i(s_i, m) = s'_i \Leftrightarrow m = t, \quad (\$)_t$$

for all $m \in M$. Let $\alpha: D(\mathbf{A}) \rightarrow M$ preserve $G \cup G_1$. If α is given by evaluation at $a \in A$ on $\alpha^{-1}(S)$, then α is also given by evaluation at a on $\alpha^{-1}(t)$.

Proof. The conclusion will follow from the First GST Lemma once we check that each $s \in \{s_1, s'_1, \dots, s_n, s'_n\}$ is in $\alpha(D(\mathbf{A}))$. Let $f \in G_1$ such that $f(t) = s$. Assume $y \in D(\mathbf{A})$ with $\alpha(y) = t$. Then $s = f(t) = f(\alpha(y)) = \alpha(f \circ y) \in \alpha(D(\mathbf{A}))$. \square

If $G = \{g_1, \dots, g_n\} \subseteq G_2$ and $S = \{s_1, s'_1, \dots, s_n, s'_n\} \subseteq M$ satisfy $(\$)_t$, then we refer to G and S as sets of *gst-functions* and *gst-elements* for t , respectively. In practice, we have found that when applying the GST Lemmas we can often choose $s'_i = s_i$. (The following theorem and the previous lemmas had a gestation which also involved joint work of the second author and B. Knox on dualisability of semigroups. The applications to semigroups will appear elsewhere.)

GST Theorem 19. Let $\underline{\mathbf{M}}$ be a finite algebra, let $S \subseteq \Sigma_{\underline{\mathbf{M}}}$ and let $G \subseteq G_2$. Assume that G includes a strong idempotent function for each $s \in S$ and that, for each element $t \in M \setminus S$, there are *gst-functions* in G and *gst-elements* in $S \cap G_1(t)$ for t . Then $\underline{\mathbf{M}}$ is dualised by $\underline{\mathbf{M}} := \langle M; G \cup G_1, R_{|S|}, \mathcal{F} \rangle$.

Proof. Let \mathbf{A} be finite and let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$. By the In- Σ Lemma 12, there is an $a \in A$ such that α is given by evaluation at a on $\alpha^{-1}(S)$. By the Second GST Lemma 18, the map α is given by evaluation at a on all of $D(\mathbf{A})$, and therefore $\underline{\mathbf{M}}$ yields a duality on \mathbf{A} . Since $\underline{\mathbf{M}}$ is of finite type, the Duality Compactness Theorem implies that $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$. \square

Under the hypotheses of the First and Second GST Lemmas, we find that α is given by evaluation at a on $\alpha^{-1}(t)$ when $(\$)_t$ holds and a is any element of $A_{\alpha, S}$. Under weaker hypotheses, α is still given by evaluation at a point in $A_{\alpha, S}$, but this point must be carefully selected.

Third GST Lemma 20. Let $\underline{\mathbf{M}}$ be a finite algebra, let $S \subseteq M$ and let $G \subseteq G_2$. Let $\alpha: D(\mathbf{A}) \rightarrow M$ and let f be a unary operation on M such that $f \upharpoonright_{\alpha(D(\mathbf{A}))} = \text{id}_{\alpha(D(\mathbf{A}))}$. Assume that, for each $t \in \alpha(D(\mathbf{A})) \setminus S$, there exist $n \in \mathbb{N}$, a set of binary homomorphisms $\{g_1, \dots, g_n\} \subseteq G$ and elements $s_1, s'_1, \dots, s_n, s'_n \in S \cap \alpha(D(\mathbf{A}))$ such that

$$\bigwedge_{i=1}^n g_i(s_i, m) = s'_i \Leftrightarrow f(m) = t \quad (\phi)_t$$

for all $m \in M$. Assume that α preserves G and that α is given by evaluation at $a \in A$ on $\alpha^{-1}(S)$. Then

- (i) $\alpha = f \circ e_A(a)$, and
- (ii) if f is a term function of $\underline{\mathbf{M}}$, then $\alpha = e_A(f^A(a))$, where f^A is the term function of \mathbf{A} corresponding to f .

Proof. (i) Let $x \in D(\mathbf{A})$. If $\alpha(x) \in S$, then

$$\alpha(x) = f(\alpha(x)) = f(x(a)) = (f \circ e_A(a))(x).$$

So we can assume that $t := \alpha(x) \notin S$. There is some $n \in \mathbb{N}$ with $g_i \in G$ and $s_i, s'_i \in S \cap \alpha(D(\mathbf{A}))$, for each $i \in \{1, \dots, n\}$, such that $(\varphi)_t$ holds. We have

$$\begin{aligned} f(\alpha(x)) = \alpha(x) = t &\Rightarrow \bigwedge_{i=1}^n g_i(s_i, \alpha(x)) = s'_i \\ &\Rightarrow \bigwedge_{i=1}^n g_i(s_i, x(a)) = s'_i \\ &\Rightarrow f(x(a)) = t = \alpha(x), \end{aligned}$$

by $(\varphi)_t$ and Lemma 16. Thus $\alpha = f \circ e_A(a)$.

(ii) If f is a term function of $\underline{\mathbf{M}}$, then we have

$$\alpha(x) = (f \circ e_A(a))(x) = f(x(a)) = x(f^A(a)) = e_A(f^A(a))(x),$$

for all $x \in D(\mathbf{A})$. \square

We end this section by presenting some applications of our results, beginning with the In- Σ Theorem 13. Applications of the Third GST Lemma will be postponed until Section 5.

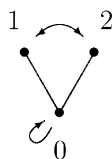
Let $M = \{0, \dots, n\}$, for some $n \in \mathbb{N}$. A unary operation $f : M \rightarrow M$ will be denoted by the string $f(0) \cdots f(n)$. A binary operation $g : M^2 \rightarrow M$ will be denoted by the matrix

$$\begin{pmatrix} g(0, n) & \cdots & g(n, n) \\ \vdots & \ddots & \vdots \\ g(0, 0) & \cdots & g(n, 0) \end{pmatrix}.$$

In the following discussion, we make claims that binary operations on a set M are homomorphisms of some unary algebra $\underline{\mathbf{M}}$. These claims can be checked by looking at diagrams like those in Figs. 2 and 3. To show that a meet operation on M is algebraic over $\underline{\mathbf{M}}$, we draw the operations of $\underline{\mathbf{M}}$ as endomorphisms of the appropriate semilattice. We show that other binary operations are algebraic by checking them against diagrams of $\underline{\mathbf{M}}^2$ and $\underline{\mathbf{M}}$.

Example 21. The unary algebras $\langle \{0, 1, 2\}; 021 \rangle$ and $\langle \{0, 1, 2\}; 021, 000 \rangle$ are both dualisable.

The binary operation
 $\wedge_0 := \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
 is algebraic over $\underline{\mathbf{M}}$.



The binary operation
 $g := \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$
 is algebraic over $\underline{\mathbf{M}}$.

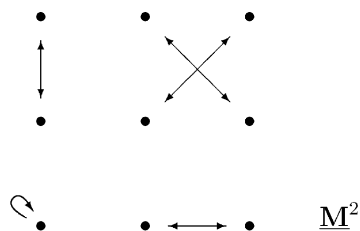


Fig. 2. $\underline{\mathbf{M}} = \langle \{0, 1, 2\}; 021 \rangle$.

Proof. Let $\underline{\mathbf{M}} = \langle \{0, 1, 2\}; 021 \rangle$ and let \wedge_0 and g be the binary homomorphisms shown in Fig. 2. Then \wedge_0 is a strong idempotent function for 1 and 2, and g is a strong idempotent function for 0. By the In- Σ Theorem 13, the algebra $\underline{\mathbf{M}}$ is dualised by $\langle \{0, 1, 2\}; \wedge_0, g, R_3, \mathcal{T} \rangle$, and therefore also by $\langle \{0, 1, 2\}; R_3, \mathcal{T} \rangle$. Since the binary operations \wedge_0 and g both preserve 0, it also follows that $\langle \{0, 1, 2\}; 021, 000 \rangle$ is dualised by $\langle \{0, 1, 2\}; R_3, \mathcal{T} \rangle$, with the appropriate shift in our interpretation of R_3 . \square

Example 21 illustrates an easy way to extend results obtained using Theorem 13. Once we have found binary operations to prove that $\Sigma_{\underline{\mathbf{M}}} = M$, we can add extra fundamental operations to the algebra, provided they preserve the binary operations, and it will remain dualisable.

The dualisability of the algebras in Example 21 also follows from a result which will appear in a follow-up paper [2] focusing exclusively on unary algebras. In this paper, we shall prove that a finite unary algebra is dualisable provided each of its operations is either constant or a permutation.

Example 22. The unary algebra $\langle \{0, 1, 2, 3\}; 0010, 0321 \rangle$ is dualisable.

Proof. Using Fig. 3, it is easy to check that the strong idempotent functions

$$\begin{pmatrix} 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}$$

are algebraic over $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3\}; 0010, 0321 \rangle$. So $\Sigma_{\underline{\mathbf{M}}} = M$, and $\underline{\mathbf{M}}$ is dualisable by the In- Σ Theorem 13. \square

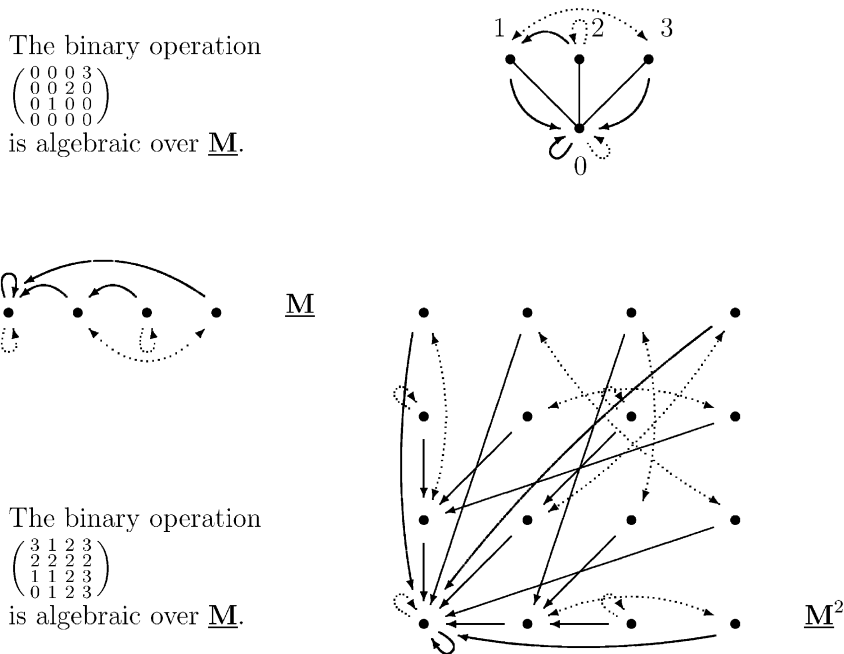


Fig. 3. $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3\}; 0010, 0321 \rangle$.

Our last example illustrates the GST Theorem 19 at work. Note that a more complicated dualising structure for this algebra can be obtained using Theorem 5, by regarding the operations as endomorphisms of the lattice $0 < 1 < 3 < 2$.

Example 23. The unary algebra $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3\}; 0010, 0011 \rangle$ is dualisable.

Proof. Define $u := 0010$ and $v := 0011$. We ask the reader to draw the appropriate diagrams to verify that the binary operations

$$g_0 := \begin{pmatrix} 3 & 3 & 2 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 3 \end{pmatrix}, \quad g_{12} := \begin{pmatrix} 0 & 0 & 3 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad g_3 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

are algebraic over $\underline{\mathbf{M}}$. The map g_0 is a strong idempotent function for 0 and g_{12} is a strong idempotent function for 1 and 2. So $S := \{0, 1, 2\} \subseteq \Sigma_{\underline{\mathbf{M}}}$ and $M \setminus S = \{3\}$. (In fact, there is no strong idempotent function for 3, so $\Sigma_{\underline{\mathbf{M}}} = \{0, 1, 2\}$.) We have $g_3(1, m) = 1 \Leftrightarrow m = 3$. Since $u \circ v = v \circ u$, we see that $v \in G_1$. It follows that $1 = v(3) \in S \cap G_1(3)$. By the GST Theorem 19, it follows that $\underline{\mathbf{M}}$ is dualised by $\langle \{0, 1, 2, 3\}; G_1 \cup \{g_0, g_{12}, g_3\}, R_3, \mathcal{F} \rangle$, and therefore also by $\langle \{0, 1, 2, 3\}; R_3, \mathcal{F} \rangle$. \square

The next two sections develop some extra tools which help us to find more applications of the GST Lemmas.

4. Dualisable term retracts

In this section, we will show that a dualisable term retract of $\underline{\mathbf{M}}$ serves exactly the same role as $\Sigma_{\underline{\mathbf{M}}}$: providing a set $S \subseteq M$ for which we can guarantee that $A_{\alpha, S} \neq \emptyset$ when $\alpha: D(\mathbf{A}) \rightarrow M$ preserves the right structure. To do this, we use a construction given in Davey [5]. In [5], it is assumed that there is an algebra $\underline{\mathbf{D}}$ which is dualised by some structure $\underline{\mathbf{D}}$, and that we have a finite algebra $\underline{\mathbf{M}} \in \mathbb{ISP}\underline{\mathbf{D}}$ such that $\underline{\mathbf{D}}$ is a retract of $\underline{\mathbf{M}}$. The structure $\underline{\mathbf{D}}$ is then used to define a structure $\underline{\mathbf{M}}$ which dualises $\underline{\mathbf{M}}$. Our next lemma deals with a similar situation. We have a dualisable algebra $\underline{\mathbf{D}}$ and a finite algebra $\underline{\mathbf{M}}$ which has $\underline{\mathbf{D}}$ as a retract via a unary term function. Using the construction given in [5], we produce a structure $\underline{\mathbf{M}}$ which may not dualise $\underline{\mathbf{M}}$, but which can help to establish a duality for \mathbb{ISPM} .

Let $\gamma: \underline{\mathbf{M}} \rightarrow \underline{\mathbf{D}}$ be both a subretraction and a unary term function. We say that γ is a *term retraction*. For each $n \in \mathbb{N}$, we use γ^n to denote the natural product map $\gamma \times \cdots \times \gamma: \underline{\mathbf{M}}^n \rightarrow \underline{\mathbf{D}}^n$. Each algebraic relation r on $\underline{\mathbf{D}}$ is also an algebraic relation on $\underline{\mathbf{M}}$. For each n -ary algebraic operation g on $\underline{\mathbf{D}}$, we can define the n -ary algebraic operation $g_{\underline{\mathbf{M}}}: = g \circ \gamma^n$ on $\underline{\mathbf{M}}$. For each n -ary algebraic partial operation h on $\underline{\mathbf{D}}$, we define the n -ary algebraic partial operation $h_{\underline{\mathbf{M}}}: = h \circ \gamma^n \upharpoonright_{(\gamma^n)^{-1}(\text{dom}(h))}$ on $\underline{\mathbf{M}}$. Let G be a set of algebraic operations on $\underline{\mathbf{D}}$ and let H be a set of algebraic partial operations on $\underline{\mathbf{D}}$. We can now define the sets $G \circ \gamma := \{g_{\underline{\mathbf{M}}} \mid g \in G\}$ and $H \circ \gamma := \{h_{\underline{\mathbf{M}}} \mid h \in H\}$.

The map γ is very useful because of its dual personality as both a term function and a homomorphism. Consider an algebra $\mathbf{A} \in \mathbb{ISPM}$. Then we can find an embedding $\varphi: \mathbf{A} \hookrightarrow \underline{\mathbf{M}}^S$ with left inverse $\psi: \varphi(\mathbf{A}) \rightarrow \mathbf{A}$. As \mathbf{A} is an algebra of the same type as $\underline{\mathbf{M}}$, the unary term function γ of $\underline{\mathbf{M}}$ has an interpretation $\gamma^{\mathbf{A}}$ on \mathbf{A} . We can also regard γ as a unary algebraic operation on $\underline{\mathbf{M}}$. In this guise, the map γ extends pointwise to the endomorphism $\gamma^S: \underline{\mathbf{M}}^S \rightarrow \underline{\mathbf{D}}^S$ of $\underline{\mathbf{M}}^S$. The fundamental operations on $\underline{\mathbf{M}}^S$ are also defined pointwise from the fundamental operations on $\underline{\mathbf{M}}$. So γ^S agrees with the unary term function $\gamma^{\underline{\mathbf{M}}^S}$. This implies that $\gamma^S \circ \varphi(A) \subseteq \varphi(A)$, whence we obtain the endomorphism $\psi \circ \gamma^S \circ \varphi: \mathbf{A} \rightarrow \mathbf{A}$. But we have

$$\psi \circ \gamma^S \circ \varphi = \psi \circ \gamma^{\underline{\mathbf{M}}^S} \circ \varphi = \psi \circ \varphi \circ \gamma^{\mathbf{A}} = \gamma^{\mathbf{A}}.$$

So the endomorphism $\psi \circ \gamma^S \circ \varphi$ agrees with the term function $\gamma^{\mathbf{A}}$. We are therefore justified in using the simpler notation $\gamma: \mathbf{A} \rightarrow \mathbf{A}$ for both maps. Throughout the next lemma, we will alter our view of γ as required.

Term Retract Lemma 24. *Let $\underline{\mathbf{M}}$ be a finite algebra, let $\gamma: \underline{\mathbf{M}} \rightarrow \underline{\mathbf{D}}$ be a term retraction and assume that $\underline{\mathbf{D}}$ is dualised by $\underline{\mathbf{D}} = \langle D; G, H, R, \mathcal{F} \rangle$. Define*

$$\underline{\mathbf{M}} := \langle M; \{\gamma\} \cup (G \circ \gamma), H \circ \gamma, R, \mathcal{F} \rangle,$$

let $\mathbf{A} \in \mathbb{ISPM}$ and let $\alpha: \mathbf{A} \rightarrow \underline{\mathbf{M}}$ be a morphism. Then α is an evaluation on $\alpha^{-1}(D)$.

Proof. Let $\mathbf{A} \in \mathbb{ISP}\mathbf{M}$ and let $\alpha: \mathbf{D}(\mathbf{A}) \rightarrow \mathbf{M}$ be a morphism. There is an embedding $\varphi: \mathbf{A} \hookrightarrow \mathbf{M}^S$ for some set S . Since γ is both a term function and a homomorphism, the algebra $\gamma \circ \varphi(\mathbf{A}) = \varphi \circ \gamma(\mathbf{A})$ is a subalgebra of \mathbf{D}^S . Thus $\gamma(\mathbf{A}) \in \mathbb{ISP}\mathbf{D}$ and we can define $\mathbf{D}'(\gamma(\mathbf{A}))$ to be the dual of $\gamma(\mathbf{A})$ with respect to \mathbf{D} .

We want to define the map $\beta: \mathbf{D}'(\gamma(\mathbf{A})) \rightarrow D$ by $\beta(y) = \alpha(y \circ \gamma)$. To see that β is well defined, let $y \in \mathbf{D}'(\gamma(\mathbf{A}))$. Then $\gamma \circ y = y$ and therefore

$$\gamma(\alpha(y \circ \gamma)) = \alpha(\gamma \circ y \circ \gamma) = \alpha(y \circ \gamma),$$

since α preserves γ . Thus $\alpha(y \circ \gamma) \in D$, whence β is well defined.

Next, we shall show that β is a \mathbf{D} -morphism. Let $h \in H$ be an n -ary partial operation on \mathbf{D} and let $y_1, \dots, y_n \in \mathbf{D}'(\gamma(\mathbf{A}))$. Then

$$\begin{aligned} (y_1, \dots, y_n) &\in \text{dom}(h)^{\mathbf{D}'(\gamma(\mathbf{A}))} \\ &\Rightarrow (\forall a \in A) (y_1(\gamma(a)), \dots, y_n(\gamma(a))) \in \text{dom}(h) \\ &\Rightarrow (\forall a \in A) \gamma^n(y_1(\gamma(a)), \dots, y_n(\gamma(a))) \in \text{dom}(h) \\ &\Rightarrow (y_1 \circ \gamma, \dots, y_n \circ \gamma) \in \text{dom}(h_{\mathbf{M}})^{\mathbf{D}(\mathbf{A})} \\ &\Rightarrow (\alpha(y_1 \circ \gamma), \dots, \alpha(y_n \circ \gamma)) \in \text{dom}(h_{\mathbf{M}}) \\ &\Rightarrow \gamma^n(\alpha(y_1 \circ \gamma), \dots, \alpha(y_n \circ \gamma)) \in \text{dom}(h) \\ &\Rightarrow (\beta(y_1), \dots, \beta(y_n)) \in \text{dom}(h). \end{aligned}$$

For all $(y_1, \dots, y_n) \in \text{dom}(h)^{\mathbf{D}'(\gamma(\mathbf{A}))}$ we have

$$\begin{aligned} \beta(h(y_1, \dots, y_n)) &= \alpha(h(y_1, \dots, y_n) \circ \gamma) \\ &= \alpha(h \circ \gamma^n(y_1 \circ \gamma, \dots, y_n \circ \gamma)) \\ &= \alpha(h_{\mathbf{M}}(y_1 \circ \gamma, \dots, y_n \circ \gamma)) \\ &= h_{\mathbf{M}}(\alpha(y_1 \circ \gamma), \dots, \alpha(y_n \circ \gamma)) \\ &= h \circ \gamma^n(\beta(y_1), \dots, \beta(y_n)) \\ &= h(\beta(y_1), \dots, \beta(y_n)). \end{aligned}$$

Therefore β preserves H . Similarly, the map β preserves G . As α is continuous and preserves R , it follows that β is a \mathbf{D} -morphism. So β is given by evaluation at some $b \in \gamma(A) \subseteq A$. We will show that α is also given by evaluation at b on $\alpha^{-1}(D)$.

Let $x \in \alpha^{-1}(D)$. For all $a \in A$ we have $x(\gamma(a)) = \gamma(x(a)) \in D$, as γ is a term function. So $x \upharpoonright_{\gamma(A)} \in \mathbf{D}'(\gamma(\mathbf{A}))$. This gives us

$$\begin{aligned} \alpha(x) &= \gamma(\alpha(x)) \quad \text{as } \alpha(x) \in D \\ &= \alpha(\gamma \circ x) \quad \text{as } \alpha \text{ preserves } \gamma \end{aligned}$$

$$\begin{aligned}
&= \alpha(x \circ \gamma) \quad \text{as } \gamma \text{ is a term function} \\
&= \alpha(x \upharpoonright_{\gamma(A)} \circ \gamma) \\
&= \beta(x \upharpoonright_{\gamma(A)}) \quad \text{as } x \upharpoonright_{\gamma(A)} \in D'(\gamma(\mathbf{A})) \\
&= x \upharpoonright_{\gamma(A)}(b) \quad \text{as } \beta = e_{\gamma(\mathbf{A})}(b) \\
&= x(b).
\end{aligned}$$

Thus α is given by evaluation at b on $\alpha^{-1}(D)$. \square

We can use the Term Retract Lemma to obtain new dualities when we team it with the First GST Lemma 17.

Term Retract Theorem 25. *Let $\underline{\mathbf{M}}$ be a finite algebra, let $\gamma: \underline{\mathbf{M}} \rightarrow \underline{\mathbf{D}}$ be a term retraction and assume that $\underline{\mathbf{D}} = \langle D; G, H, R, \mathcal{T} \rangle$ is a dualising structure for $\underline{\mathbf{D}}$ with finite type. Define*

$$\underline{\mathbf{M}} := \langle M; G_2 \cup (G \circ \gamma), H \circ \gamma, R \cup R_{|\Sigma_{\underline{\mathbf{M}}}|}, \mathcal{T} \rangle.$$

Suppose that, for all $m \in M \setminus D$ and all $t \in M \setminus \Sigma_{\underline{\mathbf{M}}}$, there exist gst-functions in G_2 and gst-elements in $\Sigma_{\underline{\mathbf{M}}} \cap G_1(m)$ for t . Then $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$.

Proof. Let $\mathbf{A} \in \mathbb{ISP}\underline{\mathbf{M}}$ be finite and let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ be a morphism. By the Term Retract Lemma, if $\alpha(D(\mathbf{A})) \subseteq D$, then α is an evaluation. So we can assume there is some $m \in \alpha(D(\mathbf{A})) \setminus D$. Since α preserves G_1 , we have $G_1(m) \subseteq \alpha(D(\mathbf{A}))$. Thus, for all $t \in \alpha(D(\mathbf{A})) \setminus \Sigma_{\underline{\mathbf{M}}}$, there is some $n \in \mathbb{N}$ with $g_i \in G_2$ and $s_i, s'_i \in \Sigma_{\underline{\mathbf{M}}} \cap \alpha(D(\mathbf{A}))$, for each $i \in \{1, \dots, n\}$, such that $(\$)_t$ holds. By the In- Σ Lemma 12 and the First GST Lemma 17, we conclude that α is an evaluation. It follows that $\underline{\mathbf{M}}$ yields a duality on $\mathbb{ISP}\underline{\mathbf{M}}$, by the Duality Compactness Theorem. \square

This result is easily modified to cover the case where there is a finite number of term retractions onto dualisable subalgebras.

Multi Term Retract Theorem 26. *Let $\underline{\mathbf{M}}$ be a finite algebra. Assume that, the map $\gamma_i: \underline{\mathbf{M}} \rightarrow \underline{\mathbf{D}}_i$ is a term retraction and that $\underline{\mathbf{D}}_i = \langle D; G^{(i)}, H^{(i)}, R^{(i)}, \mathcal{T} \rangle$ is a dualising structure for $\underline{\mathbf{D}}_i$ with finite type for each $i \in \{1, \dots, k\}$. Define*

$$\underline{\mathbf{M}} := \langle M; G_2 \cup \bigcup_{i=1}^k (G^{(i)} \circ \gamma_i), \bigcup_{i=1}^k (H^{(i)} \circ \gamma_i), \bigcup_{i=1}^k R^{(i)} \cup R_{|\Sigma_{\underline{\mathbf{M}}}|}, \mathcal{T} \rangle.$$

Assume that, for all transversals T of $\{M \setminus D_i \mid i \in \{1, \dots, k\}\}$ and all $t \in M \setminus \Sigma_{\underline{\mathbf{M}}}$, there exist gst-functions in G_2 and gst-elements in $\Sigma_{\underline{\mathbf{M}}} \cap G_1(T)$ for t . Then $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$.

Proof. This is an easy generalisation of the proof of the Term Retract Theorem. \square

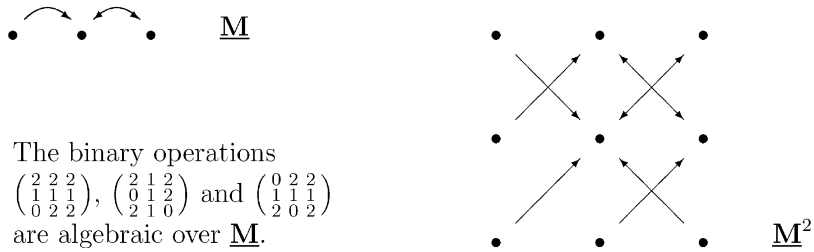


Fig. 4. $\underline{\mathbf{M}} = \langle \{0, 1, 2\}; 121 \rangle$.

The following example illustrates the Term Retract Theorem 25. In [12], the Term Retract Theorem is used to help prove that every finite unary algebra with only one fundamental operation is dualisable.

Example 27. The unary algebras $\langle \{0, 1, 2\}; 121 \rangle$ and $\langle \{0, 1, 2\}; 121, 111, 222 \rangle$ are dualisable.

Proof. Let $\underline{\mathbf{M}}$ be either $\langle \{0, 1, 2\}; 121 \rangle$ or $\langle \{0, 1, 2\}; 121, 111, 222 \rangle$ and define $\underline{\mathbf{M}} = \langle \{0, 1, 2\}; G_3, R_1, \mathcal{S} \rangle$. The map $212 = 121 \circ 121$ is a term retraction from $\underline{\mathbf{M}}$ onto the subalgebra $\underline{\mathbf{D}}$ with underlying set $D := \{1, 2\}$. The algebra $\underline{\mathbf{D}}$ is either a 2-element set with involution or a 2-element doubly-pointed set with involution. It is shown in Clark and Davey [1] (see Table 10.2) that, in either case, $\underline{\mathbf{D}}$ is dualised by its ternary homomorphisms. We will use Theorem 25 to prove that $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$.

The binary operations

$$g_0 := \begin{pmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}, \quad g_1 := \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 2 \\ 2 & 1 & 0 \end{pmatrix} \quad \text{and} \quad g_2 := \begin{pmatrix} 0 & 2 & 2 \\ 1 & 1 & 1 \\ 2 & 0 & 2 \end{pmatrix}$$

are all algebraic over $\underline{\mathbf{M}}$. To see this, refer to Fig. 4. The function g_0 is a strong idempotent function for 0. We have

$$g_1(0, m) = 0 \Leftrightarrow m = 1 \quad \text{and} \quad g_2(0, m) = 0 \Leftrightarrow m = 2,$$

for all $m \in M$. Applying Theorem 25, we conclude that $\underline{\mathbf{M}}$ is dualised by $\underline{\mathbf{M}}$. \square

We can build on the previous example to find another dualisable algebra.

Example 28. The unary algebra $\langle \{0, 1, 2, 3\}; 1212, 0121 \rangle$ is dualisable.

Proof. The map 0121 is term retraction from $\underline{\mathbf{M}} := \langle \{0, 1, 2, 3\}; 1212, 0121 \rangle$ onto the algebra $\underline{\mathbf{D}} := \langle \{0, 1, 2\}; 121, 012 \rangle$. It follows from Example 27 that $\underline{\mathbf{D}}$ is dualisable. The

binary operations

$$g_{03} := \begin{pmatrix} 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \end{pmatrix}, \quad g_1 := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 \end{pmatrix} \quad \text{and} \quad g_2 := \begin{pmatrix} 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 3 \\ 2 & 2 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

are algebraic over $\underline{\mathbf{M}}$. So $\{0, 3\} \subseteq \Sigma_{\underline{\mathbf{M}}}$. The map g_1 is a gst-function for 1 using the gst-element 3, and g_2 is a gst-function for 2 using the gst-element 3. Thus $\underline{\mathbf{M}}$ is dualisable by Theorem 25. \square

As an illustration of the Multi Term Retract Theorem 26, we give a second dualising structure for the algebra $\langle \{0, 1, 2\}; 010, 002 \rangle$ of lattice endomorphisms.

Example 29. The unary algebra $\langle \{0, 1, 2\}; 010, 002 \rangle$ is dualisable.

Proof. Define $\underline{\mathbf{M}} = \langle \{0, 1, 2\}; 010, 002 \rangle$. Then 010 is a term retraction from $\underline{\mathbf{M}}$ onto the subalgebra $\underline{\mathbf{D}}_1$ of $\underline{\mathbf{M}}$ with $D_1 := \{0, 1\}$, and 002 is a term retraction from $\underline{\mathbf{M}}$ onto the subalgebra $\underline{\mathbf{D}}_2$ of $\underline{\mathbf{M}}$ with $D_2 := \{0, 2\}$. Since $\underline{\mathbf{D}}_1$ and $\underline{\mathbf{D}}_2$ are both pointed sets, they are dualised by total structures of finite type. (See [1]. In fact, their dualisability follows from either Theorem 5 or the In- Σ Theorem 13 as well.) Now consider the binary operations

$$\wedge_0 := \begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad *_0 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix}$$

on M . The reader can either verify that \wedge_0 and $*_0$ are algebraic over $\underline{\mathbf{M}}$ or obtain this fact from Lemma 30 in the next section. As \wedge_0 is a strong idempotent function for 1 and 2, we have $\{1, 2\} \subseteq \Sigma_{\underline{\mathbf{M}}}$. We now apply the Multi Term Retract Theorem 26. Observe that $T = \{1, 2\}$ is the only transversal of the complements of D_1 and D_2 , and that 0 is the only element that may be in $M \setminus \Sigma_{\underline{\mathbf{M}}}$. For each $m \in M$, we have

$$1 *_0 m = 1 \quad \& \quad 2 *_0 m = 2 \quad \Leftrightarrow \quad m = 0,$$

so $*_0$ is a gst-function for 0, using the gst-elements $1, 2 \in \Sigma_{\underline{\mathbf{M}}} \cap T$. It now follows from Theorem 26 that $\underline{\mathbf{M}}$ is dualisable. \square

5. Flat-semilattice and near-projection operations

Relative to a fixed element 0 of a finite set M , we define two complementary binary operations $\wedge_0 : M^2 \rightarrow M$ and $*_0 : M^2 \rightarrow M$ by

$$k \wedge_0 l = \begin{cases} k & \text{if } k = l, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad k *_0 l = \begin{cases} 0 & \text{if } k = l, \\ k & \text{otherwise.} \end{cases}$$

The 0-flat-semilattice operation \wedge_0 on M is a strong idempotent function for every element of M except 0, while the 0-near-projection operation $*_0$ on M satisfies the condition

$$\&_{s \in M \setminus \{0\}} s *_0 m = s \iff m = 0, \tag{$)_0$$

for all $m \in M$. Conveniently, it is very easy to say when \wedge_0 and $*_0$ are algebraic if $\underline{\mathbf{M}}$ is a unary algebra. A function $f : M \rightarrow M$ is said to be one-to-one away from 0 if $f(k) = f(l) \neq 0$ implies $k = l$, for all $k, l \in M$.

Lemma 30. Assume that $0 \in M$ and let $f : M \rightarrow M$.

- (i) The map f preserves \wedge_0 if and only if either f is constant or f preserves 0 and is one-to-one away from 0.
- (ii) The map f preserves $*_0$ if and only if f preserves 0 and f is one-to-one away from 0.

In particular, if f preserves $*_0$, then f also preserves \wedge_0 .

Proof. Since f preserves \wedge_0 says precisely that f is an endomorphism of the 0-flat semilattice $\langle M; \wedge_0 \rangle$, item (i) is a very easy semilattice calculation and so is omitted.

Assume that f preserves $*_0$. Then $f(0) = f(0 *_0 0) = f(0) *_0 f(0) = 0$. Let $k, l \in M$ with $k \neq l$ and $f(k) = f(l)$. We have $f(k) = f(k *_0 l) = f(k) *_0 f(l) = 0$. Finally, assume that f preserves 0 and is one-to-one away from 0. If $k, l \in M$ with $k \neq l$, then $f(k *_0 k) = f(0) = 0 = f(k) *_0 f(k)$ and $f(k *_0 l) = f(k) = f(k) *_0 f(l)$, so f preserves $*_0$. Thus (ii) holds. \square

Let $\underline{\mathbf{M}}$ be a finite algebra with $0 \in M$ whose operations all preserve \wedge_0 and $*_0$. Define $\underline{\mathbf{M}} := \langle M; \wedge_0, *_0, R_n, \mathcal{T} \rangle$, where $n := |M| - 1$. Let \mathbf{A} be finite and let $\alpha : D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ be a surjective morphism. Since $M \setminus \{0\} \subseteq \Sigma_{\underline{\mathbf{M}}}$, the gst-elements in $(\$)_0$ are all in $\Sigma_{\underline{\mathbf{M}}} \cap \alpha(D(\mathbf{A}))$. So we can use the In- Σ Lemma 12 and the First GST Lemma 17 to conclude that α is an evaluation. Our goal is to find conditions under which $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$. The problem will be that α is not in general surjective, so our choice of gst-elements will be restricted. We begin with an example for which this restriction is easy to overcome. In the next section, we will use other techniques to extend this example.

Example 31. The endomorphisms of a prime cyclic group yield a dualisable unary algebra. More precisely, for each prime p , the unary algebra $\underline{\mathbf{M}} := \langle \mathbb{Z}_p; \text{End}(\mathbb{Z}_p) \rangle$ is dualised by $\underline{\mathbf{M}} := \langle \mathbb{Z}_p; \wedge_0, *_0, +, R_{p-1}, \mathcal{T} \rangle$.

Proof. Let \mathbf{A} be finite and let $\alpha : D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ be a morphism. If α is surjective then, as above, it is an evaluation. The set $\alpha(D(\mathbf{A}))$ determines a subgroup of \mathbb{Z}_p . So it remains to consider the case $\alpha(D(\mathbf{A})) = \{0\}$, for which α is given by evaluation at the constant $0^{\mathbf{A}}$. By the Duality Compactness Theorem, $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$. \square

The following lemma tells us, in this context, by exactly what margin a morphism $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ can possibly fail to be an evaluation if it is not surjective. For each subset N of M , we define the map $f_N: M \rightarrow M$ by

$$f_N(k) = \begin{cases} k & \text{if } k \in N, \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 32. *Let $\underline{\mathbf{M}}$ be a finite algebra with $0 \in M$. Assume that \wedge_0 and $*_0$ are algebraic over $\underline{\mathbf{M}}$ and define $\underline{\mathbf{M}} := \langle M; \wedge_0, *_0, R_n, \mathcal{F} \rangle$, where $n := |M| - 1$. Let \mathbf{A} be finite, let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ be a morphism and define $N := \alpha(D(\mathbf{A}))$. Then there is an $a \in A$ such that*

- (i) $\alpha = f_N \circ e_{\mathbf{A}}(a)$, and
- (ii) $\alpha = e_{\mathbf{A}}(f_N^{\mathbf{A}}(a))$, if f_N is a term function of $\underline{\mathbf{M}}$.

Proof. Since $\wedge_0 \in G_2$, we have $M \setminus \{0\} \subseteq \Sigma_{\underline{\mathbf{M}}}$. If $N = \{0\}$, then, for any $a \in A$ and $x \in D(\mathbf{A})$, we have $(f_N \circ e_{\mathbf{A}}(a))(x) = f_{\{0\}}(x(a)) = 0 = \alpha(x)$. We will therefore assume that $N \setminus \{0\}$ is non-empty. By the In- Σ Lemma 12, there is some $a \in A$ such that α is given by evaluation at a on $\alpha^{-1}(N \setminus \{0\})$. We have

$$\begin{aligned} \bigwedge_{s \in N \setminus \{0\}} s *_0 m = s &\Leftrightarrow \bigwedge_{s \in N \setminus \{0\}} m \neq s \\ &\Leftrightarrow m = 0 \text{ or } m \notin N \\ &\Leftrightarrow f_N(m) = 0, \end{aligned}$$

for all $m \in M$. It follows from the Third GST Lemma 20 that $\alpha = f_N \circ e_{\mathbf{A}}(a)$ and that if f_N is a term function of $\underline{\mathbf{M}}$, then $\alpha = e_{\mathbf{A}}(f_N^{\mathbf{A}}(a))$. \square

We use $\text{Pol}(\wedge_0, *_0)$ to denote the clone of operations on M that preserve \wedge_0 and $*_0$. It follows from Lemma 30 that $f_N \in \text{Pol}(\wedge_0, *_0)$, for each $N \subseteq M$.

Theorem 33. *Let $\underline{\mathbf{M}}$ be a finite algebra with $0 \in M$. If*

$$\{f_{M \setminus \{\ell\}} \mid \ell \in M \setminus \{0\}\} \subseteq \text{Clo}(\underline{\mathbf{M}}) \subseteq \text{Pol}(\wedge_0, *_0),$$

*then $\underline{\mathbf{M}}$ is dualised by $\underline{\mathbf{M}} := \langle M; \wedge_0, *_0, R_n, \mathcal{F} \rangle$, where $n := |M| - 1$.*

Proof. Since $f_{M \setminus \{\ell\}} \in \text{Clo}(\underline{\mathbf{M}})$, for all $\ell \in M$, and $f_P \circ f_Q = f_{P \cap Q}$, for all $P, Q \subseteq M$, it follows easily that $f_N \in \text{Clo}(\underline{\mathbf{M}})$, for all $N \subseteq M$. Let \mathbf{A} be finite and let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ be a morphism. By Lemma 32(ii), there is some $b \in A$ such that $\alpha = e_{\mathbf{A}}(b)$. So $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$ by the Duality Compactness Theorem. \square

As immediate consequences of Lemma 30 and Theorem 33, we have the following examples of dualisable unary algebras.

Example 34. The unary algebra $\langle \{0, 1, 2, 3\}; \mu, \nu, \omega \rangle$, illustrated in Fig. 5, is dualised by $\langle \{0, 1, 2, 3\}; R_3, \mathcal{F} \rangle$.

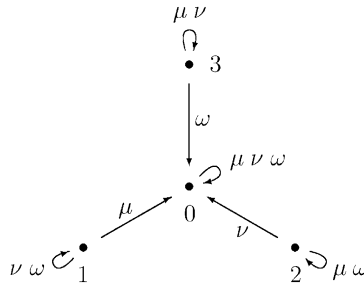


Fig. 5. $\langle \{0, 1, 2, 3\}; \mu, \nu, \omega \rangle$

Example 35. The unary algebra $\langle \{0, 1, 2\}; 001, 020 \rangle$ is dualisable.

Proof. Define $\underline{\mathbf{M}} = \langle \{0, 1, 2\}; 001, 020 \rangle$. Since 001 and 020 preserve 0 and are one-to-one away from 0, they preserve the operations \wedge_0 and $*_0$ by Lemma 30. Since $001 \circ 020 = 010$ and $020 \circ 001 = 002$, we have

$$\{010, 002\} \subseteq \text{Clo}(\underline{\mathbf{M}}) \subseteq \text{Pol}(\wedge_0, *_0),$$

whence $\underline{\mathbf{M}}$ is dualised by $\underline{\mathbf{M}} := \langle \{0, 1, 2\}; \wedge_0, *_0, R_2, \mathcal{F} \rangle$ by Theorem 33. \square

Example 36. Let $n \in \mathbb{N}$ and let F denote the set of all unary functions on $M = \{0, \dots, n\}$ which fix 0 and are one-to-one away from 0. Then the algebra $\underline{\mathbf{M}} = \langle M; F \rangle$ is dualised by $\underline{\mathbf{M}} := \langle M; \wedge_0, *_0, R_n, \mathcal{F} \rangle$.

Some of the theorems in this paper can be used to establish a strong form of dualisability. We will say that a finite algebra $\underline{\mathbf{M}}$ is *ideally dualisable* if $\underline{\mathbf{M}}$ is dualisable and every term reduct of $\underline{\mathbf{M}}$ is dualisable. So an ideally dualisable algebra $\underline{\mathbf{M}}$ determines a principal ideal of dualisable clones in the lattice of clones on M . Theorem 5, the In- Σ Theorem 13 and the GST Theorem 19 can all be used to prove ideal dualisability. The Term Retract Theorem 25 and Theorem 33 are not of this type. The algebra $\langle \{0, 1, 2\}; 001, 020 \rangle$, of Example 35, was shown to be dualisable using Theorem 33. In a subsequent paper [2], we will show that the term reduct $\langle \{0, 1, 2\}; 001, 010 \rangle$ of $\langle \{0, 1, 2\}; 001, 020 \rangle$ is not dualisable.

Example 34 is reminiscent of the algebra $\underline{\mathbf{M}}$ illustrated in Fig. 1. The algebra $\underline{\mathbf{M}}$ is interesting because it defies all of the methods we have presented, as the patient reader can verify. Its operations do not preserve any lattice order on M so Theorem 5 does not apply, and 0 is not a strong idempotent of any binary homomorphism so the In- Σ Theorem 13 does not apply. By Lemma 30 we have $\Sigma_{\underline{\mathbf{M}}} = \{1, 2, 3\}$, but the GST Theorem 19 cannot help us with 0 as $\Sigma_{\underline{\mathbf{M}}} \cap G_1(0) = \{1, 2, 3\} \cap \{0\} = \emptyset$. It is tempting to try to apply the Multi Term Retract Theorem 26 using the three term retracts $u(\underline{\mathbf{M}})$, $v(\underline{\mathbf{M}})$ and $w(\underline{\mathbf{M}})$, but this fails as there are not enough gst-functions to obtain $(\$)_0$ using only the gst-elements 1 and 2. This leaves us with Lemma 32, but if $N = \{0, 1, 2\}$,

for example, then f_N is not a term function of $\underline{\mathbf{M}}$. With considerable extra effort, we show that it is still possible to prove that $\alpha = e_{\mathbf{A}}(f_N^{\mathbf{A}}(a))$ in this special case.

Theorem 37. *Let $M = \{0, 1, 2, 3\}$ and define*

$$F' := \{f_{\{1\}}, f_{\{2\}}, f_{\{3\}}\} \quad \text{and} \quad F'' := \{f_N \mid N \subseteq M\}.$$

If $F' \subseteq F \subseteq F''$, then $\underline{\mathbf{M}} = \langle M; F \rangle$ is dualised by $\underline{\mathbf{M}} = \langle M; R_3, \mathcal{F} \rangle$.

Proof. Define $u := f_{\{1\}}$, $v := f_{\{2\}}$ and $w := f_{\{3\}}$. Let $n \in \mathbb{N}$, let \mathbf{A} be a subalgebra of $\underline{\mathbf{M}}^n$ and let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ be a morphism. Then α preserves the operations \wedge_0 and $*_0$. The map $f_{\{0\}} := u \circ v$ is a term function of $\underline{\mathbf{M}}$. So $\bar{0} := (0, \dots, 0) \in A$ and $x(\bar{0}) = x(f_{\{0\}}(\bar{0})) = f_{\{0\}}(x(\bar{0})) = 0$, for all $x \in D(\mathbf{A})$.

Define $N := \alpha(D(\mathbf{A}))$. By Lemma 32, there is an $a \in A$ with $\alpha = f_N \circ e_{\mathbf{A}}(a)$. If f_N is a term function of $\underline{\mathbf{M}}$, then $\alpha = e_{\mathbf{A}}(f_N^{\mathbf{A}}(a))$ is an evaluation.

Assume, therefore, that f_N is not a term function of $\underline{\mathbf{M}}$. We must have $0 \in N$, since

$$0 = \alpha(x) *_0 \alpha(x) = \alpha(x *_0 x) \in \alpha(D(\mathbf{A})) = N$$

for all $x \in D(\mathbf{A})$. As f_N is not a term function of $\underline{\mathbf{M}}$, this implies that $N = \{0, 1, 2\}$, $N = \{0, 1, 3\}$ or $N = \{0, 2, 3\}$. By symmetry, we can assume that $N = \{0, 1, 2\}$.

Define the maps $x_1, x_2, x_3: A \rightarrow M$ by

$$\begin{aligned} x_1(c) &= \begin{cases} 1 & \text{if } c^{-1}(1) = a^{-1}(1), \\ 0 & \text{otherwise,} \end{cases} & x_2(c) &= \begin{cases} 2 & \text{if } c^{-1}(2) = a^{-1}(2), \\ 0 & \text{otherwise,} \end{cases} \\ x_3(c) &= \begin{cases} 3 & \text{if } c^{-1}(3) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Let $x \in D(\mathbf{A})$ with $\alpha(x) = 1$. Then $1 = \alpha(x) = f_N(x(a)) = u(x(a)) = x(u(a))$. Since $x(\bar{0}) = 0$, we have $u(a) \neq \bar{0}$ and so $a^{-1}(1) \neq \emptyset$. Similarly, $a^{-1}(2) \neq \emptyset$. Using these facts, it is straightforward to check that x_1 , x_2 and x_3 preserve the unary operations in F'' , and therefore belong to $D(\mathbf{A})$. Since f_N is not a term function of $\underline{\mathbf{M}}$, the relation $r := M^3 \setminus \{(1, 2, 0)\}$ is algebraic over $\underline{\mathbf{M}}$, and therefore α preserves r . We have

$$(\alpha(x_1), \alpha(x_2), \alpha(x_3)) = (f_N(x_1(a)), f_N(x_2(a)), f_N(x_3(a))) = (1, 2, 0) \notin r,$$

so there is some $b \in A$ such that $(x_1(b), x_2(b), x_3(b)) \notin r$. This implies that $(x_1(b), x_2(b), x_3(b)) = (1, 2, 0)$, giving $b^{-1}(1) = a^{-1}(1)$, $b^{-1}(2) = a^{-1}(2)$ and $b^{-1}(3) = \emptyset$. Therefore $b = f_N(a)$.

We have shown that $f_N(a) = b \in A$ in this example, even though f_N is not a term function of $\underline{\mathbf{M}}$. It remains to prove that $\alpha = e_{\mathbf{A}}(b)$. To do this, let $y \in D(\mathbf{A})$. Since $b^{-1}(3) = \emptyset$, we have

$$w(y(b)) = y(w(b)) = y(\bar{0}) = 0$$

and therefore $y(b) \neq 3$. If $y(b) = 1$, then

$$u(y(a)) = y(u(a)) = y(u(b)) = u(y(b)) = 1,$$

so $y(a) = 1$ and $\alpha(y) = f_N(y(a)) = f_N(1) = 1 = y(b)$. Similarly, if $y(b) = 2$, then $\alpha(y) = y(b)$. Finally, if $y(b) = 0$, then we get

$$u(y(a)) = y(u(a)) = y(u(b)) = u(y(b)) = 0$$

and

$$v(y(a)) = y(v(a)) = y(v(b)) = v(y(b)) = 0,$$

which implies that $y(a) \in \{0, 3\}$ and gives us $\alpha(y) = f_N(y(a)) = 0 = y(b)$. Thus, $\alpha = e_A(b)$ and, by the Duality Compactness Theorem, we conclude that $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$. \square

Finally we have a proof of the dualisability of the algebra illustrated in Fig. 1, as this is the special case in which $F = F'$.

Example 38. The unary algebra $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3\}; u, v, w \rangle$, shown in Fig. 1, is dualised by $\underline{\mathbf{M}} = \langle \{0, 1, 2, 3\}; R_3, \mathcal{F} \rangle$.

6. Group operations

Under the right circumstances, a binary homomorphism which is also a group operation plays a central role in producing a duality. This stems from the fact that the product of two classes of a group congruence is equal to, rather than just contained in, another congruence class. We give two proofs of the following lemma, one exploiting this fact and another applying the First GST Lemma 17.

Lemma 39. Let $\underline{\mathbf{M}}$ be a finite algebra which has a binary homomorphism $*$ such that $\langle M; * \rangle$ is a group. Assume that $\alpha: D(\mathbf{A}) \rightarrow M$ preserves $*$ and that $a \in A$. Define

$$S_a := \{s \in \alpha(D(\mathbf{A})) \mid \alpha \upharpoonright_{\alpha^{-1}(s)} = e_A(a) \upharpoonright_{\alpha^{-1}(s)}\}.$$

If $S_a \neq \emptyset$, then $\langle S_a; * \rangle$ is a subgroup of $\langle M; * \rangle$.

Proof 1. We show that S_a is closed under $*$. Let $r, s \in S_a$ and let $x, y \in D(\mathbf{A})$ such that $\alpha(x) = r$ and $\alpha(y) = s$. Then $r * s = \alpha(x * y) \in \alpha(D(\mathbf{A}))$. Let $z \in D(\mathbf{A})$ with $\alpha(z) = r * s$ and define K to be the kernel of α . Then $\alpha(z) = r * s = \alpha(x * y)$ and $z * K = (x * y) * K = (x * K) * (y * K)$. Choose $x' \in x * K$ and $y' \in y * K$ such that $z = x' * y'$. Since $\alpha(x') = \alpha(x) = r \in S_a$, we have $r = \alpha(x') = x'(a)$. Similarly, $s = \alpha(y') = y'(a)$. Thus

$$\alpha(z) = r * s = x'(a) * y'(a) = (x' * y')(a) = z(a),$$

showing that $r * s \in S_a$. \square

Proof 2. We again show that S_a is closed under $*$. Let $r, s \in S_a \subseteq \alpha(D(\mathbf{A}))$ and define $t := r * s$. Then $r^{-1} \in S_a$, since α preserves $^{-1}$. For all $m \in M$, we have

$$r^{-1} * m = s \iff m = t.$$

So $r * s = t \in S_a$ by the First GST Lemma 17. \square

Group Lemma 40. *Let $\underline{\mathbf{M}}$ be a finite algebra which has a binary homomorphism $*$ such that $\langle M; * \rangle$ is a group. Let $\alpha: D(\mathbf{A}) \rightarrow M$ preserve $*$. Assume that $S \neq \emptyset$ generates $\alpha(D(\mathbf{A}))$ as a subgroup of $\langle M; * \rangle$ and that α is an evaluation on $\alpha^{-1}(S)$. Then α is an evaluation.*

Proof. There is some $a \in A$ such that α is given by evaluation at a on $\alpha^{-1}(S)$. Thus $S \subseteq S_a \subseteq \alpha(D(\mathbf{A}))$, and $S_a = \alpha(D(\mathbf{A}))$ by Lemma 39. \square

We will prove a theorem providing a significant class of dualisable unary algebras, using the Term Retract Lemma 24 and the Group Lemma. If we begin with a finite group $\mathbf{M}_0 = \langle M; * \rangle$, we can ask whether the unary algebra $\underline{\mathbf{M}} = \langle M; \text{End}(\mathbf{M}_0) \rangle$ is dualisable; in particular, whether we can augment the structure $\underline{\mathbf{M}}_0 = \langle M; *, \mathcal{T} \rangle$ so that it will yield a duality on $\mathbb{I}\mathbb{S}\mathbb{P}\underline{\mathbf{M}}$. The general problem of determining for which groups \mathbf{M}_0 the algebra $\underline{\mathbf{M}} = \langle M; \text{End}(\mathbf{M}_0) \rangle$ is dualisable remains open and challenging. Our next theorem shows that $\underline{\mathbf{M}}$ is dualisable when the group \mathbf{M}_0 is cyclic.

Let $k \in \mathbb{N}$, let p_1, \dots, p_k be distinct primes and let $r_1, \dots, r_k \in \mathbb{N}$. Define the cyclic group $\mathbf{M}_0 := \mathbb{Z}_{p_1^{r_1}} \times \dots \times \mathbb{Z}_{p_k^{r_k}}$ of order $n := p_1^{r_1} \dots p_k^{r_k}$. For each $s \in \mathbb{Z}_n$, let $A_s: \mathbf{M}_0 \rightarrow \mathbf{M}_0$ denote the endomorphism of \mathbf{M}_0 given by $A_s(a) = sa$. Then $\text{End}(\mathbf{M}_0) = \{A_s \mid s \in \mathbb{Z}_n\}$. The endomorphisms of \mathbf{M}_0 are products of endomorphisms of the factor groups. For each $i \in \{1, \dots, k\}$ and $t \in \mathbb{Z}_{p_i^{r_i}}$, define the endomorphism $\lambda_t: \mathbb{Z}_{p_i^{r_i}} \rightarrow \mathbb{Z}_{p_i^{r_i}}$ by $\lambda_t(b) = tb$. Let $s \in \mathbb{Z}_n$ and, for each $i \in \{1, \dots, k\}$, define $s_i := s \pmod{p_i^{r_i}}$. Then $A_s = \lambda_{s_1} \times \dots \times \lambda_{s_k}$, where

$$\lambda_{s_1} \times \dots \times \lambda_{s_k}(a_1, \dots, a_k) := (\lambda_{s_1}(a_1), \dots, \lambda_{s_k}(a_k)).$$

Thus

$$\text{End}(\mathbf{M}_0) = \{ \lambda_{s_1} \times \dots \times \lambda_{s_k} \mid s_i \in \mathbb{Z}_{p_i^{r_i}} \text{ for all } i \in \{1, \dots, k\} \}.$$

Theorem 41. *The endomorphisms of a finite cyclic group yield a dualisable unary algebra. More precisely, for each $n \in \mathbb{N}$, the unary algebra $\underline{\mathbf{M}} := \langle \mathbb{Z}_n; \text{End}(\mathbb{Z}_n) \rangle$ of endomorphisms of the cyclic group \mathbb{Z}_n is dualised by $\underline{\mathbf{M}} := \langle \mathbb{Z}_n; +, g, R_1, \mathcal{T} \rangle$, for some binary homomorphism g of $\underline{\mathbf{M}}$.*

Proof. We will argue by induction on the number of prime factors of the order of the cyclic group. The induction commences by observing that the endomorphisms of \mathbb{Z}_1 yield a dualisable unary algebra.

Let $k \in \mathbb{N}$, let p_1, \dots, p_k be distinct primes and let $r_1, \dots, r_k \in \mathbb{N}$. Define the group $\mathbf{M}_0 := \mathbb{Z}_{p_1^{r_1}} \times \dots \times \mathbb{Z}_{p_k^{r_k}}$ and the algebra $\underline{\mathbf{M}} := \langle M; \text{End}(\mathbf{M}_0) \rangle$, where $M = \mathbb{Z}_{p_1^{r_1}} \times \dots \times \mathbb{Z}_{p_k^{r_k}}$. It is straightforward to show that, for each $i \in \{1, \dots, k\}$, the map $g_i: \mathbb{Z}_{p_i^{r_i}}^2 \rightarrow \mathbb{Z}_{p_i^{r_i}}$, given by

$$g_i(a, b) = \begin{cases} a & \text{if } na = nb = p_i^{r_i-1} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise,} \end{cases}$$

preserves the endomorphisms of the group $\mathbb{Z}_{p_i^{r_i}}$. So we can define the binary homomorphism $g: \underline{\mathbf{M}}^2 \rightarrow \underline{\mathbf{M}}$ by $g(a, b)(i) = g_i(a(i), b(i))$. We will show that $\underline{\mathbf{M}}$ is dualised by $\underline{\mathbf{M}} := \langle M; +, g, R_1, \mathcal{F} \rangle$.

Let $\mathbf{A} \in \mathbb{I}\mathbb{S}\mathbb{P}\mathbf{M}$ be finite and let $\alpha: D(\mathbf{A}) \rightarrow \underline{\mathbf{M}}$ be a morphism. Since α preserves $+$, it must also preserve each endomorphism of the cyclic group \mathbf{M}_0 . Let $m \in M$ be a generator of $\alpha(D(\mathbf{A}))$ as a subgroup of \mathbf{M}_0 and choose some $z \in D(\mathbf{A})$ such that $\alpha(z) = m$.

Case (i): $m(j) = 0$ for some $j \in \{1, \dots, k\}$. The endomorphism

$$u := \lambda_1 \times \dots \times \lambda_1 \times \lambda_0 \times \lambda_1 \times \dots \times \lambda_1$$

of \mathbf{M}_0 is a term retraction of $\underline{\mathbf{M}}$ such that $\alpha(D(\mathbf{A})) \subseteq u(M)$. We are assuming, for the induction, that $u(\underline{\mathbf{M}})$ is dualised by $\langle u(M); + \upharpoonright_{u(M)}, g \upharpoonright_{u(M)}, R_1 \upharpoonright_{u(M)}, \mathcal{F} \rangle$. The map α preserves the structure on $\langle M; u, + \circ u^2, g \circ u^2, R_1 \upharpoonright_{u(M)}, \mathcal{F} \rangle$, since it preserves the endomorphism u . By the Term Retract Lemma 24, it follows that α is an evaluation.

Case (ii): $m(i) \neq 0$ for all $i \in \{1, \dots, k\}$. By the Group Lemma, it suffices to show that α is an evaluation on $\alpha^{-1}(m)$. It is easy to check that the map g is a strong idempotent function for the element $s \in M$ given by $s(i) = p_i^{r_i-1}$. For each $i \in \{1, \dots, k\}$, we have $m(i) \neq 0$, and so there is an endomorphism v_i of $\mathbb{Z}_{p_i^{r_i}}$ such that $v_i(m(i)) = p_i^{r_i-1}$. Define the endomorphism $v := v_1 \times \dots \times v_k$ of \mathbf{M}_0 . Then $v(m) = s$. We have $s = v(m) = v(\alpha(z)) = \alpha(v \circ z) \in \alpha(D(\mathbf{A}))$. Since α also preserves R_1 , it follows from the In- Σ Lemma 12 that there exists $a \in A_{z,s}$. Thus $s = \alpha(v \circ z) = v \circ z(a)$.

Let $i \in \{1, \dots, k\}$. Since $v \circ z(a) = s = v(m)$, we have $v_i(z(a)(i)) = p_i^{r_i-1} = v_i(m(i))$. It follows that $z(a)(i)$ and $m(i)$ have the same order as elements of the cyclic group $\mathbb{Z}_{p_i^{r_i}}$ and therefore generate the same subgroup of $\mathbb{Z}_{p_i^{r_i}}$. So there must be an endomorphism w_i of $\mathbb{Z}_{p_i^{r_i}}$ such that $w_i(z(a)(i)) = m(i)$. Define $w := w_1 \times \dots \times w_k$. This gives us

$$z(w(a))(i) = w(z(a))(i) = w_i(z(a)(i)) = m(i),$$

so that $z(w(a)) = m$.

We claim that $w(a) \in A_{z,s}$. To see this, let $x \in \alpha^{-1}(s)$. Then

$$x(w(a)) = w(x(a)) = w(s) = w(v \circ z(a)) = v(z(w(a))) = v(m) = s.$$

So $w(a) \in A_{z,s}$.

We will finish the proof by showing that $w(a) \in A_{z,m}$. Let $y \in \alpha^{-1}(m)$. Then

$$\alpha(v \circ z - z + y) = v(\alpha(z)) - \alpha(z) + \alpha(y) = v(m) - m + m = s.$$

Since $w(a) \in A_{z,s}$, we have

$$\begin{aligned} s &= (v \circ z - z + y)(w(a)) = v \circ z(w(a)) - z(w(a)) + y(w(a)) \\ &= v(m) - m + y(w(a)) = s - m + y(w(a)), \end{aligned}$$

which implies that $y(w(a)) = m$. Thus $w(a) \in A_{z,m}$ and α is given by evaluation on $\alpha^{-1}(m)$. By the Group Lemma, it follows that α is an evaluation. Hence $\underline{\mathbf{M}}$ dualises $\underline{\mathbf{M}}$, by the Duality Compactness Theorem. \square

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