

THE AXIOMATIZABILITY OF TOPOLOGICAL PREVARIETIES

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ABSTRACT. We investigate first-order axiomatic descriptions of naturally occurring classes of Boolean topological structures (these structures can have operations and relations, and carry a compatible compact Hausdorff topology with a basis of clopen sets). Our methods utilize inverse limits and ultraproducts of finite structures. We illustrate the range of possible axiomatizations of these classes with applications of our methods to Boolean topological lattices, graphs, ordered structures, unary algebras and semigroups. For example, whereas the class of all k -colorable graphs is known to be definable by universal Horn sentences, we find the class of continuously k -colorable Boolean topological graphs is not even first-order definable.

Motivation. The original motivation for this study comes from the theory of natural dualities, which is a tool for studying finitely generated quasivarieties of algebras. A natural duality provides us with a dual equivalence between the quasivariety generated by a finite algebra and a special dual category of compact topological structures. In order to utilize a natural duality fully, it is generally necessary that it be accompanied by an effective description of the objects in the dual category. This dual category is generated by some finite discretely topologized structure \mathbf{M} , being the class $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ of all isomorphic copies of topologically closed substructures of non-zero direct powers (with the product topology) of \mathbf{M} .

This study began with the goal of finding a logical description of the objects in the dual category $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$. In fact, there is an obvious candidate for such a description. Assume that $\mathbf{M} = \langle M; G, R \rangle$ has (total) operations G , relations R , and implicitly carries the discrete topology. It is easy to check that each member of $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ is a **Boolean topological structure**, that is, a structure $\mathfrak{X} = \langle X; G, R, \mathcal{T} \rangle$ carrying a Boolean topology \mathcal{T} (compact, Hausdorff and with a basis of clopen sets) such that each operation in G is continuous and each relation in R is topologically closed. Moreover, each member of $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ also belongs, as a non-topological structure, to the class $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$ generated by \mathbf{M} with no topology. We know that $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$ is described by the universal Horn theory of \mathbf{M} :

$$\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M}) = \text{Mod}(\text{Th}_{\text{uH}}(\mathbf{M})).$$

Thus every member of $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ is a Boolean topological model of the universal Horn theory of \mathbf{M} :

$$\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M}) \subseteq \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{M})).$$

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Following Clark and Krauss [11], we call $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ a **topological quasivariety**, and ask if $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ is exactly the class of all Boolean topological models of the universal Horn theory of \mathbf{M} , that is, if $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{M}))$.

Consider, for example, the quasivariety generated by the ring (with identity) of integers modulo 6. Clark and Davey [8] found a natural duality for this quasivariety by choosing as the generator of the dual category $\mathbf{Z}_6 := \langle \{0, 1, 2, 3, 4, 5\}; * \rangle$, with the binary operation $*$ defined by $a \equiv_2 a * b \equiv_3 b$. This generator is a rectangular band, that is, a model of the universal Horn theory of anti-commutative semigroups:

$$\Sigma := \{(x * y) * z \approx x * (y * z), x * y \approx y * x \implies x \approx y\}.$$

In fact [8], the dual category $\mathbb{IS}_c\mathbb{P}^+(\mathbf{Z}_6)$ is exactly the class of Boolean topological rectangular bands. Thus $\mathbb{IS}_c\mathbb{P}^+(\mathbf{Z}_6) = \text{Mod}_{\text{Bt}}(\Sigma) = \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{Z}_6))$.

Is it true that $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{M}))$, for every finite structure \mathbf{M} ? Unfortunately not, as was first demonstrated by Stralka [35]. Under Priestley's duality [33], the quasivariety of bounded distributive lattices is dual to the topological quasivariety $\mathbb{IS}_c\mathbb{P}^+(\mathbf{2})$ of **Priestley spaces**, generated by the 2-element chain $\mathbf{2} := \langle \{0, 1\}; \leq \rangle$. Stralka exhibited a Boolean space with a topologically closed order that is not a Priestley space, that is, he gave a member of $\text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{2}))$ that is not in $\mathbb{IS}_c\mathbb{P}^+(\mathbf{2})$ (Example 6.1).

Stralka's example led Clark, Davey, Haviar, Pitkethly and Talukder [10] to designate a finite structure \mathbf{M} as **standard** if the topological quasivariety $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ is exactly the class of Boolean topological models of the universal Horn theory of \mathbf{M} ; in symbols, the structure \mathbf{M} is standard if

$$\mathbb{IS}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{M})).$$

Their paper [10] introduces this notion, explores its basic properties, and gives many examples of standard and non-standard structures. Standardness of algebras in particular ($R = \emptyset$) has been further studied by Clark, Davey, Freese and Jackson [9], who established a general condition that guarantees the standardness of a host of familiar finite algebras. Our Theorem 2.13 extends this result to allow relations as well.

Our terminology is motivated by an early attempt to give a logical description of $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$. Clark and Krauss [11] showed that *every* topological quasivariety is definable as the class of all compact models of a collection of sentences, with the formal structure of universal Horn sentences, in an infinitary language capable of describing topological properties. Thus a topological quasivariety is standard if it is definable via *standard* universal Horn sentences. While their result is conceptually interesting, because it applies to every topological quasivariety, it has had little impact due to the complexity of the required language.

Although a finite structure \mathbf{M} may not be standard, we may still ask if $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ is definable among all Boolean topological structures of its type via first-order sentences. We say that $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ is **first-order definable** if it is exactly the class of Boolean topological models of some set Σ of first-order sentences, that is, if

$$\mathbb{IS}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\Sigma).$$

In Sections 3 and 5, we develop a number of general techniques, involving inverse limits and ultraproducts, that allow us to show that many topological quasivarieties are *not* first-order definable. In particular, we will show that the class of Priestley spaces is not first-order definable (Example 6.2).

Our preference for describing the topological quasivariety $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ via universal Horn sentences stems from the fact that $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$ (without topology) is a universal Horn class. This bias is partly justified by Corollary 2.18: if $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\Sigma)$, for some set Σ of *universal* sentences, then indeed $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{M}))$. However, we find in Section 4 that every finite structure of finite type \mathbf{M} that has a lattice reduct generates a first-order-definable topological quasivariety $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\Sigma)$, and then we give a specific example of a lattice \mathbf{M}_{3-3} for which Σ cannot be taken to be $\text{Th}_{\text{uH}}(\mathbf{M}_{3-3})$. Thus \mathbf{M}_{3-3} generates a first-order-definable topological quasivariety but is not standard.

As illustrated by \mathbf{Z}_6 , \mathbf{M}_{3-3} and $\mathbf{2}$, each finite structure \mathbf{M} can be assigned one of three types, according to the axiomatizability of the topological quasivariety that it generates:

- (i) $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{M}))$, that is, the structure \mathbf{M} is standard;
- (ii) $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\Sigma)$, for some set Σ of first-order sentences but for no set Σ of universal sentences;
- (iii) $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ is not first-order definable at all.

The primary theme of this study will be methods for classifying finite structures into these three types. While our methods will prove powerful enough to produce many interesting examples of each type, they fall far short of providing a complete characterization of the structures having any one of these types.

Note that, in general, the theory of natural dualities allows for partial operations in the type of a generator of a dual category. However, our techniques do not work smoothly in the presence of partial operations, and so we shall restrict our structures to operations and relations; see Remark 2.5.

More generally. Compatible Boolean topologies arise naturally in several different contexts. Trivially, a *finite structure* carries the discrete topology, which is Boolean, and this topology gives rise to the compatible Boolean topologies on the members of the topological quasivariety it generates. Finite discrete structures also give rise to compatible Boolean topologies on structures that are *profinite*, that is, structures that are inverse limits of finite structures. The topology on a profinite structure is the relative product topology, which is always Boolean. The *Cantor space*, obtained from the unit interval by removing middle thirds, is a Boolean space isomorphic to the ω power of the 2-element discrete space.

Experience shows that structures that admit a compatible compact Hausdorff topology, and in particular a compatible Boolean topology, often behave in important ways like finite structures. For instance, whereas an infinite lattice does not necessarily have greatest and least elements, a compact Hausdorff topological lattice does (Lemma 4.1). Graph colorings provide another example. For any cardinal κ , the complete graph on κ vertices is a simple graph that cannot be colored with fewer than κ colors. In contrast, we will see that every Boolean topological simple graph can be colored with finitely many colors (Theorem 7.3).

The study of the dual categories arising from natural dualities extends to a much broader context. Notice that the class of all Boolean topological structures of a fixed type is closed under the operators \mathbb{I} , \mathbb{S}_c and \mathbb{P}^+ . Borrowing the terminology used for non-topological algebras, we define a **topological prevariety** to be any class of Boolean topological structures that is closed under \mathbb{I} , \mathbb{S}_c and \mathbb{P}^+ . For example, the smallest topological prevariety containing a given class \mathcal{K} of Boolean topological

structures is $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{K})$. For each Boolean topological structure $\mathbf{X} = \langle X; G, R, \mathcal{T} \rangle$, we refer to $\mathbf{X} := \langle X; G, R \rangle$ as the **non-topological reduct** of \mathbf{X} . For each class \mathcal{N} of non-topological structures, we use \mathcal{N}_{Bt} to denote the class of all Boolean topological structures with non-topological reduct in \mathcal{N} .

The quest to understand a topological quasivariety $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ naturally leads us to the associated universal Horn class $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$. It turns out that the problems we have posed and the methods we will develop to solve them apply equally well if we begin with an arbitrary universal Horn class (**uH-class**) \mathcal{M} . We shall be interested in those Boolean topological structures whose non-topological reduct is in \mathcal{M} .

In general, we would not expect a structure in \mathcal{M} to admit a compatible Boolean topology unless that topology arises in some special way from the construction of its underlying set. For example, each structure in the class \mathcal{M}_{fin} of finite members of \mathcal{M} carries the discrete topology. Thus each member of the topological prevariety $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}})$ has its non-topological reduct in \mathcal{M} and carries a compatible Boolean topology, and is therefore in \mathcal{M}_{Bt} . We call this topological prevariety the **Boolean core** of \mathcal{M} , and denote it by

$$\mathcal{M}_{\text{Bc}} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}}) \subseteq \mathcal{M}_{\text{Bt}}.$$

For example, if \mathbf{M} is a finite structure generating the uH-class $\mathcal{M} := \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$, then we have $\mathcal{M}_{\text{Bc}} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}}) = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$, which is the topological prevariety generated by \mathbf{M} .

The definition $\mathcal{M}_{\text{Bc}} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}})$ gives us a structural (or semantic) description of \mathcal{M}_{Bc} . We would like to augment this description with a logical (or syntactic) description of \mathcal{M}_{Bc} . Extending our previous definition, we say that \mathcal{M}_{Bc} is **first-order definable** if it consists of all Boolean topological models of some set Σ of first-order sentences, that is, if $\mathcal{M}_{\text{Bc}} = \text{Mod}_{\text{Bt}}(\Sigma)$. We will say that \mathcal{M}_{Bc} is **universally definable** or **Horn definable** if Σ can be taken to be a set of universal or universal Horn sentences, respectively. Our ideal scenario is to be able to take Σ to be $\text{Th}_{\text{uH}}(\mathcal{M})$ for, in that case, we have $\mathcal{M}_{\text{Bc}} = \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathcal{M})) = \mathcal{M}_{\text{Bt}}$, which is the class of *all* Boolean topological structures with non-topological reduct in \mathcal{M} . We say that the uH-class \mathcal{M} is **standard** if $\mathcal{M}_{\text{Bc}} = \mathcal{M}_{\text{Bt}}$, that is, if *every* Boolean topological structure with non-topological reduct in \mathcal{M} arises in this standard way as a closed substructure of a product of finite members of \mathcal{M} . Note that a finite structure \mathbf{M} is standard (by the original definition) if and only if the uH-class $\mathcal{M} := \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$ is standard (by our new definition), since $\mathcal{M}_{\text{Bc}} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$.

It is perhaps counter-intuitive that, even if a uH-class \mathcal{M} is not generated by its finite members ($\mathcal{M} \neq \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathcal{M}_{\text{fin}})$), the uH-class \mathcal{M} may yet be standard, meaning that the corresponding topological class \mathcal{M}_{Bt} *is* generated by finite structures ($\mathcal{M}_{\text{Bt}} = \mathcal{M}_{\text{Bc}} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}})$). The class of all simple graphs is an example of such a uH-class (Section 7).

In this paper, we exhibit examples of standardness and of non-standardness, but we will primarily concentrate on techniques for showing that a uH-class \mathcal{M} is *not* standard. These techniques will use either an inverse limit or an ultraproduct to construct a Boolean topological structure \mathbf{X} that is guaranteed to be in \mathcal{M}_{Bt} but not in \mathcal{M}_{Bc} . Often we will be able to construct \mathbf{X} to be a model of the universal Horn, universal or first-order theory of \mathcal{M}_{Bc} ; thus showing, in addition to non-standardness, that \mathcal{M}_{Bc} is not Horn, universally or even first-order definable.

For a uH-class \mathcal{M} , we have $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}}) \subseteq [\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}} \subseteq \mathcal{M}_{\text{Bt}}$, and \mathcal{M} is standard if all three classes are equal. When the first two classes are equal,

$\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}}) = [\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}}$, we say that \mathcal{M} is **pre-standard**. We will see that standardness, pre-standardness, Horn definability and universal definability are all equivalent if \mathcal{M} is generated by a set of finite structures (Proposition 2.17). We will also give examples to show that, in general, standardness is stronger than Horn definability, universal definability and pre-standardness, and that these three conditions are still stronger than first-order definability.

Of course, definability can be considered for any topological prevariety, not just those generated by a single finite structure or by all the finite members of a uH-class. Starting from any class \mathcal{K} of Boolean topological structures, we can ask whether the class $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{K})$ is first-order definable, universally definable or Horn definable. In this paper, we shall not be working in this more general setting. Here, we are interested in classes of Boolean topological structures that are based on familiar, well-behaved classes of non-topological structures.

Summary. We will give the necessary background in Section 1, and develop fundamental properties of standardness and definability in Section 2. Our techniques for proving non-standardness are developed in Sections 3 and 5. Starting with a uH-class \mathcal{M} , Section 3 gives three techniques for constructing an inverse limit of finite structures that is in \mathcal{M}_{Bt} but not in \mathcal{M}_{Bc} . Each of these techniques will allow us to conclude that \mathcal{M}_{Bc} is not even universally definable. Our preferred method is the Second Inverse Limit Technique 3.9, since it has the added bonus of showing that the original uH-class \mathcal{M} is not finitely axiomatizable. In Section 5, we present two techniques for constructing an ultraproduct of finite structures that will give us a Boolean topological structure that is a model of the first-order theory of \mathcal{M}_{Bc} but is not in \mathcal{M}_{Bc} . When this can be done, we are guaranteed that \mathcal{M}_{Bc} is not first-order definable. The remainder of the paper consists of applications of these techniques.

The generators of the natural dual categories for distributive lattices, De Morgan algebras, median algebras, Stone and double Stone algebras and Kleene algebras are all non-standard [10]. In Section 6, we use these facts and the First Ultraproduct Technique 5.2 to show that none of these dual categories is first-order definable.

In Section 7, we study the class \mathcal{G} of simple graphs, defined by the universal Horn formulæ $x \approx x$ and $x \sim y \implies y \sim x$. We will prove that \mathcal{G} is standard, and that every Boolean topological simple graph is continuously colorable using finitely many colors. In contrast, we find, for each $k \geq 2$, that the uH-class of k -colorable graphs is not standard: there is a Boolean topological graph that is k -colorable but not continuously k -colorable. It will then follow from the Second Ultraproduct Technique 5.3 that the topological prevariety of all continuously k -colorable graphs is not first-order definable.

Bestsennyĭ [4] gave a complete description of all 3-element unary algebras whose uH-theory is not finitely based. In Section 8, we will use the Second Inverse Limit Technique 3.9 to show that none of these unary algebras is standard, and then apply this fact and the Second Ultraproduct Technique 5.3 to show that none of the topological prevarieties they generate is even first-order definable.

These problems appear to get rapidly more difficult in the presence of non-unary operations. Nevertheless, in Section 9 we will give a complete solution for a restricted class of semigroups. We show that a finite cyclic semigroup is standard if and only if it generates a first-order-definable topological prevariety, if and only if it is finitely uH-based, if and only if it has index at most 2.

In the last section, we list several open problems. Among them we point out an apparent but as yet unproven connection between the standardness of a finite structure and the finite axiomatizability of the uH-class that it generates.

1. BACKGROUND

For us, a **structure** is of the form $\mathbf{X} = \langle X; G^{\mathbf{X}}, R^{\mathbf{X}} \rangle$, a set equipped with a set of finitary (total) operations and a set of finitary relations. The default type of a structure will always be G for the set of operation symbols and R for the set of relation symbols. A **topological structure**, $\underline{\mathbf{X}} = \langle X; G^{\underline{\mathbf{X}}}, R^{\underline{\mathbf{X}}}, \mathcal{T} \rangle$, is a structure equipped with a compatible topology \mathcal{T} : the operations are continuous and the relations are closed. We use bold font with a ‘twiddle’ for names of topological structures, and just bold font for (non-topological) structures. So, given a topological structure $\underline{\mathbf{X}}$, its non-topological reduct is denoted by \mathbf{X} . However, we shall not distinguish between the topological and non-topological versions of *finite* structures; every finite structure \mathbf{X} is understood to carry the discrete topology implicitly.

Now let $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ be topological structures. We write $\varphi : \underline{\mathbf{X}} \rightarrow \underline{\mathbf{Y}}$ to mean that φ is a continuous homomorphism, or **morphism** for short. A (non-topological) homomorphism $\varphi : \mathbf{X} \rightarrow \mathbf{Y}$ is **strong** if, for each m -ary relation $r \in R$ and each $(a_1, \dots, a_m) \in X^m$ with $(\varphi(a_1), \dots, \varphi(a_m)) \in r^{\mathbf{Y}}$, there are $b_1, \dots, b_m \in X$ such that $\varphi(b_i) = \varphi(a_i)$, for $i = 1, 2, \dots, m$, and $(b_1, \dots, b_m) \in r^{\mathbf{X}}$. A morphism $\varphi : \underline{\mathbf{X}} \rightarrow \underline{\mathbf{Y}}$ is an **isomorphism** if it is a strong, closed bijection. Recall that a morphism $\varphi : \underline{\mathbf{X}} \rightarrow \underline{\mathbf{Y}}$ is closed whenever $\underline{\mathbf{X}}$ is compact and $\underline{\mathbf{Y}}$ is Hausdorff.

A congruence θ on the algebraic reduct $\langle X; G^{\mathbf{X}} \rangle$ of \mathbf{X} determines a quotient structure \mathbf{X}/θ : for each $r \in R$, we have $(a_1/\theta, \dots, a_m/\theta) \in r^{\mathbf{X}/\theta}$ provided there are $b_1, \dots, b_m \in X$ such that $(a_i, b_i) \in \theta$, for $i = 1, 2, \dots, m$, and $(b_1, \dots, b_m) \in r^{\mathbf{X}}$. It follows that the natural homomorphism $\eta : \mathbf{X} \rightarrow \mathbf{X}/\theta$ is always strong. Moreover, if $\varphi : \mathbf{X} \rightarrow \mathbf{Y}$ is surjection, then there is a natural homomorphism $\mu : \mathbf{X}/\ker(\varphi) \rightarrow \mathbf{Y}$, which is an isomorphism if and only if φ is strong. Thus, in the presence of relations, quotients correspond exactly to strong homomorphic images. The congruence θ on \mathbf{X} is a **clopen congruence** on $\underline{\mathbf{X}}$ if each θ -class is clopen in $\underline{\mathbf{X}}$; equivalently, if θ is clopen in $\underline{\mathbf{X}}^2$. Note that, if $\underline{\mathbf{X}}$ is compact, then every clopen congruence on $\underline{\mathbf{X}}$ must have finite index.

The zero power $\underline{\mathbf{X}}^\emptyset$ is the 1-element structure with underlying set $\{\emptyset\}$ and every relation non-empty. So $\underline{\mathbf{X}}^\emptyset = \underline{\mathbf{Y}}^\emptyset$, and we denote this structure by $\mathbf{1}$. The operator \mathbb{P}^+ specifically excludes $\mathbf{1}$. For a class \mathcal{K} of Boolean topological structures, the structure $\mathbf{1}$ belongs to $\mathbb{IS}_c\mathbb{P}^+(\mathcal{K})$ if and only if it embeds into a member of \mathcal{K} .

A **universal Horn formula (uH-formula)** is a quantifier-free formula of one of the forms

$$(*) \quad \alpha \text{ or } \bigwedge \{\beta_i \mid i \in I\} \implies \alpha \text{ or } \bigvee \{\neg\beta_i \mid i \in I\},$$

where α and each β_i are atomic formulæ and I is a finite non-empty set. A class \mathcal{M} of structures is a **universal Horn class (uH-class)** if \mathcal{M} consists of all structures universally satisfying some set Σ of uH-formulæ; in symbols, if $\mathcal{M} = \text{Mod}(\Sigma)$. Given a class \mathcal{M} of structures, we denote by $\text{Th}_{\text{uH}}(\mathcal{M})$ the set of all uH-formulæ that hold universally in every member of \mathcal{M} , and by $\text{Th}_\forall(\mathcal{M})$ the set of all quantifier-free formulæ that hold universally in every member of \mathcal{M} . For a finite set \mathcal{K} of finite structures, the **uH-class generated** by \mathcal{K} is $\text{Mod}(\text{Th}_{\text{uH}}(\mathcal{K})) = \mathbb{ISP}^+(\mathcal{K})$. (See, for example, [23, 2.3.1] and [8, 1.3.4].)

A uH-class \mathcal{M} is **finitely axiomatizable** if there exists a finite set Σ of first-order sentences such that $\mathcal{M} = \text{Mod}(\Sigma)$. The Compactness Theorem tells us that this is equivalent to the existence of a finite set Σ' of uH-formulae such that $\mathcal{M} = \text{Mod}(\Sigma')$. We say that a finite structure \mathbf{M} is **finitely uH-based** if the uH-class $\mathbb{ISP}^+(\mathbf{M})$ that it generates is finitely axiomatizable.

A **quasi-atomic formula** is a uH-formula of either the first or second type in (*) above. We will use terminology for q-logic parallel to that for uH-logic. Although this paper is concerned with uH-logic, it will be necessary for us to reference the literature on q-logic, so we take a moment to review the connection.

Let \mathcal{K} be a finite set of finite structures. Then the **quasivariety generated** by \mathcal{K} is $\text{Mod}(\text{Th}_q(\mathcal{K})) = \mathbb{ISP}(\mathcal{K})$. Note that $\mathbb{ISP}(\mathcal{K}) = \mathbb{ISP}^+(\mathcal{K}) \cup \mathbb{I}(\mathbf{1})$, and consequently $\mathbb{ISP}(\mathcal{K}) = \mathbb{ISP}^+(\mathcal{K})$ if and only if $\mathbf{1}$ embeds into some member of \mathcal{K} . We will use the following fact without further reference.

Proposition 1.1. *Let $\mathbf{M} = \langle M; G^M, R^M \rangle$ be a finite structure, with R finite. Then \mathbf{M} is finitely uH-based if and only if it is finitely q-based.*

Proof. We can assume that $\mathbb{ISP}(\mathbf{M}) \neq \mathbb{ISP}^+(\mathbf{M})$, and so $\mathbf{1} \notin \mathbb{ISP}^+(\mathbf{M})$. Define the finite set of atomic formulae

$$\Gamma := \{x \approx y\} \cup \{(x, x, \dots, x) \in r \mid r \in R\},$$

which asserts that a structure is isomorphic to $\mathbf{1}$.

First assume that Σ is a finite q-axiomatization for the quasivariety $\mathbb{ISP}(\mathbf{M})$. Then $\Sigma \cup \{\neg\gamma \mid \gamma \in \Gamma\}$ is a finite uH-axiomatization for the uH-class $\mathbb{ISP}^+(\mathbf{M})$.

Now assume that Σ is a finite uH-axiomatization for $\mathbb{ISP}^+(\mathbf{M})$, and let x and y be variables not occurring in Σ . We can obtain a finite q-axiomatization for $\mathbb{ISP}(\mathbf{M})$ from Σ , by replacing each uH-formula $\bigvee\{\neg\beta_i \mid i \in I\}$ in Σ with the (finite) set of all q-formulae $\bigwedge\{\beta_i \mid i \in I\} \Rightarrow \gamma$ such that $\gamma \in \Gamma$. \square

We shall occasionally be looking at atomic classes of structures, which correspond to varieties for algebras. For a class \mathcal{K} of structures, let $\text{Th}_{\text{at}}(\mathcal{K})$ denote the set of all atomic formulae that are universally satisfied by every member of \mathcal{K} . The **atomic class generated** by \mathcal{K} is $\text{At}(\mathcal{K}) := \text{Mod}(\text{Th}_{\text{at}}(\mathcal{K})) = \mathbb{HSP}(\mathcal{K})$, where \mathbb{H} denotes ‘(not necessarily strong) homomorphic images’; see [23, 2.1.12].

We will make frequent use of two fundamental facts about the topological prevariety generated by a class \mathcal{K} of Boolean topological structures, both of which are straightforward to verify. (For a guide to the proofs, see the restricted versions of the results in [8, 1.4.3 and 1.4.4].) The first result gives necessary logical and topological conditions for a structure to belong to $\mathbb{IS}_c\mathbb{P}^+(\mathcal{K})$.

Preservation Theorem 1.2. *For each class \mathcal{K} of Boolean topological structures, we have $\mathbb{IS}_c\mathbb{P}^+(\mathcal{K}) \subseteq \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathcal{K}))$.*

The second gives necessary and sufficient ‘algebraic’ conditions for a structure to belong to $\mathbb{IS}_c\mathbb{P}^+(\mathcal{K})$.

Separation Theorem 1.3. *Let \mathcal{K} be a class of Boolean topological structures, and let \mathbf{X} be a compact topological structure with $\mathbf{X} \not\cong \mathbf{1}$. The following are equivalent:*

- (i) $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+(\mathcal{K})$;
- (ii) for each m -ary $r \in R \cup \{=\}$ and $(a_1, \dots, a_m) \in X^m \setminus r^{\mathbf{X}}$, there are $\mathbf{Y} \in \mathcal{K}$ and $\varphi : \mathbf{X} \rightarrow \mathbf{Y}$ such that $(\varphi(a_1), \dots, \varphi(a_m)) \notin r^{\mathbf{Y}}$.

In the special case that $\mathfrak{X} \cong \mathbf{1}$, we observe that condition (i) may fail but that condition (ii) is vacuously true.

As inverse limits are central to this study, we briefly review this construction. Let $\mathcal{S} = \langle \mathcal{S}; \leq \rangle$ be a directed ordered set, so that $\mathcal{S} \neq \emptyset$ and each pair of elements in \mathcal{S} has an upper bound in \mathcal{S} . Assume that, for all $i \in \mathcal{S}$, we have a topological structure \mathfrak{X}_i and, for all $j \geq i$ in \mathcal{S} , we have a morphism $\varphi_{ji} : \mathfrak{X}_j \rightarrow \mathfrak{X}_i$. Assume further that, for all $i, j, k \in \mathcal{S}$, we have

$$\varphi_{ii} = \text{id}_{X_i} \quad \text{and} \quad k \geq j \geq i \implies \varphi_{ji} \circ \varphi_{kj} = \varphi_{ki}.$$

Such a system of structures and connecting morphisms is called an **inverse system**. The **inverse limit** of this inverse system is

$$\varprojlim \{\mathfrak{X}_i \mid i \in \mathcal{S}\} := \{a \in \prod_{i \in \mathcal{S}} X_i \mid (\forall i, j \in \mathcal{S})(j \geq i \implies \varphi_{ji}(a_j) = a_i)\}.$$

Using the following lemma, we can endow the inverse limit with the relative structure and topology from the product $\prod_{i \in \mathcal{S}} \mathfrak{X}_i$.

Lemma 1.4. *Let $\{\mathfrak{X}_i \mid i \in \mathcal{S}\}$ be an inverse system of topological structures.*

- (i) *The inverse limit $\varprojlim \{\mathfrak{X}_i \mid i \in \mathcal{S}\}$ forms a substructure of $\prod \{\mathfrak{X}_i \mid i \in \mathcal{S}\}$.*
- (ii) *If each topological structure in $\{\mathfrak{X}_i \mid i \in \mathcal{S}\}$ is Hausdorff, then the inverse limit $\varprojlim \{\mathfrak{X}_i \mid i \in \mathcal{S}\}$ is topologically closed in the product $\prod \{\mathfrak{X}_i \mid i \in \mathcal{S}\}$.*
- (iii) *If each topological structure in $\{\mathfrak{X}_i \mid i \in \mathcal{S}\}$ is non-empty, compact and Hausdorff, then the inverse limit $\varprojlim \{\mathfrak{X}_i \mid i \in \mathcal{S}\}$ is non-empty.*

Proof. The proof of (i) is easy. We sketch the proof of (ii) and (iii). First assume that each \mathfrak{X}_i is Hausdorff. For each $k \in \mathcal{S}$, define the set

$$A_k := \{a \in \prod_{i \in \mathcal{S}} X_i \mid (\forall i \leq k) a_i = \varphi_{ki}(a_k)\}.$$

Each A_k is topologically closed in $\prod_{i \in \mathcal{S}} \mathfrak{X}_i$, and $\varprojlim \{\mathfrak{X}_i \mid i \in \mathcal{S}\} = \bigcap_{k \in \mathcal{S}} A_k$. So the inverse limit is topologically closed. Now assume further that each \mathfrak{X}_i is non-empty and compact. Since \mathcal{S} is directed and each \mathfrak{X}_i is non-empty, every finite intersection of members of $\{A_k \mid k \in \mathcal{S}\}$ is non-empty. By compactness, the set $\bigcap_{k \in \mathcal{S}} A_k$ is non-empty. \square

This lemma tells us, in particular, that an inverse limit of finite (discrete) structures is a Boolean topological structure.

A Boolean topological structure is said to be **profinite** if it is isomorphic to an inverse limit of finite structures. If it happens that a uH-class \mathcal{M} is closed under the formation of finite quotients (for example, if it is a variety of algebras), then every profinite member of \mathcal{M}_{Bt} is an inverse limit of finite members of \mathcal{M} . But, in general, it is possible to have profinite members of \mathcal{M}_{Bt} that are not inverse limits of finite members of \mathcal{M} itself. This phenomenon will be illustrated in Example 2.7 and many times later, and will provide us with a basis for demonstrating non-standardness.

Given a class \mathcal{K} of topological structures and a class \mathcal{N} of non-topological structures, we denote by

$$\text{Inv}(\mathcal{K}) \quad \text{and} \quad \mathcal{N}_{\text{pro}},$$

respectively, the class of (isomorphic copies of) inverse limits of members of \mathcal{K} and the class of profinite structures whose non-topological reduct is in \mathcal{N} .

The central theme of [9] is a method for establishing that the uH-class generated by a single finite algebra is standard. We will review this technique here, and show in the next section that it generalizes to our present setting. Let θ be an equivalence

on a structure $\mathbf{X} = \langle X; G^{\mathbf{X}}, R^{\mathbf{X}} \rangle$, and let F be a set of terms of type G with a distinguished variable x . We define a new relation θ_F on X by setting $(a, b) \in \theta_F$ if, for every term $t(x, \vec{y}) \in F$ and every sequence \vec{c} from X , we have $(t(a, \vec{c}), t(b, \vec{c})) \in \theta$. If $x \in F$, then θ_F is easily seen to be an equivalence on X contained in θ . If F is the set of all terms of type G with distinguished variable x , then θ_F is in fact a congruence on \mathbf{X} ; moreover, it is the largest congruence on \mathbf{X} contained in θ . This is known as the **syntactic congruence** of θ , and is denoted by $\text{Syn}(\theta)$. In general, we say that the set F **determines syntactic congruences** on \mathbf{X} if, for every equivalence θ on X , the equivalence θ_F is the syntactic congruence of θ .

The key applications of syntactic congruences depend on the following result. This result goes back at least to Numakura [30], who proved it for semigroups and distributive lattices, and it was proved in varying degrees of generality by Bergman [3], Choe [7] and Day [21] (see also Almeida and Weil [1] and Johnstone [27]). A full proof is given in [9, 4.2].

Lemma 1.5. *Let $\underline{\mathbf{X}}$ be a Boolean topological structure, and assume that there is a finite set of terms that determines syntactic congruences on \mathbf{X} . Then $\text{Syn}(\theta)$ is a clopen congruence on $\underline{\mathbf{X}}$, for every clopen equivalence θ on $\underline{\mathbf{X}}$.*

We say that a set of terms **determines syntactic congruences** in a class \mathcal{M} of structures if it determines syntactic congruences on each member of \mathcal{M} . When a finite set of terms determines syntactic congruences in \mathcal{M} , we say that \mathcal{M} has **finitely determined syntactic congruences (FDSC)**. For example, the set of terms $\{x, yx, xy, yxz\}$ determines syntactic congruences in the variety of semigroups, which therefore has FDSC [9, 5.6]. A complete system of rules to tell if a set of terms determines syntactic congruences in a variety is given in [9].

2. INVERSE LIMITS, STANDARDNESS AND DEFINABILITY

In this section, we dissect the notion of standardness, by first identifying and then analyzing four subsidiary conditions that together constitute standardness. We begin with some basic results concerning inverse limits and profiniteness.

Lemma 2.1. *Let $\underline{\mathbf{X}} = \varprojlim \{\underline{\mathbf{X}}_i \mid i \in \mathcal{S}\}$ be an inverse limit of non-empty compact Hausdorff topological structures.*

- (i) *The connecting morphisms $\varphi_{ji} : \underline{\mathbf{X}}_j \rightarrow \underline{\mathbf{X}}_i$ are all surjective if and only if the projections $\pi_i : \underline{\mathbf{X}} \rightarrow \underline{\mathbf{X}}_i$ are all surjective.*
- (ii) *There exists an inverse system $\{\underline{\mathbf{Y}}_i \mid i \in \mathcal{S}\}$ such that*
 - (a) *$\underline{\mathbf{Y}}_i$ is a topologically closed substructure of $\underline{\mathbf{X}}_i$, for each $i \in \mathcal{S}$,*
 - (b) *the inverse limit $\underline{\mathbf{Y}} := \varprojlim \{\underline{\mathbf{Y}}_i \mid i \in \mathcal{S}\}$ is equal to $\underline{\mathbf{X}}$, and*
 - (c) *the projection $\pi_i : \underline{\mathbf{Y}} \rightarrow \underline{\mathbf{Y}}_i$ is surjective, for each $i \in \mathcal{S}$.*

Proof. We only sketch the proof, which is fairly straightforward but requires a few calculations. First, for (i), it is easy to check that surjective projections imply surjective connecting morphisms.

Now assume that all the connecting morphisms are surjective, and let $j \in \mathcal{S}$. To show that $\pi_j : \underline{\mathbf{X}} \rightarrow \underline{\mathbf{X}}_j$ is surjective, let $b \in X_j$. For each $k \geq j$ in \mathcal{S} , define

$$A_k := \{a \in \prod_{i \in \mathcal{S}} X_i \mid a_j = b \text{ and } (\forall i \leq k) a_i = \varphi_{ki}(a_k)\}.$$

As in the proof of Lemma 1.4, we can show that $\emptyset \neq \bigcap_{k \geq j} A_k \subseteq X$. Now choose any $a \in \bigcap_{k \geq j} A_k$. We have $\pi_j(a) = a_j = b$, whence π_j is surjective. So (i) holds.

For (ii), we define \mathbf{Y}_i to be the substructure of \mathbf{X}_i on the set $Y_i := \pi_i(X)$, with the relative topology. It is straightforward to check that these \mathbf{Y}_i 's form an inverse system of topological structures that fulfils the requirements of (ii). \square

We shall say that an inverse system is **surjective** if all of its connecting morphisms are surjective, and we shall say that an inverse limit is **surjective** if all of its projections are surjective. By our previous lemma, an inverse limit is surjective if and only if the inverse system is surjective, and every inverse limit can be obtained as the inverse limit of some surjective inverse system.

Our next lemma gives a method for identifying profinite structures.

Lemma 2.2. *Let \mathbf{X} be a compact topological structure, and assume that there is a set \mathcal{S} of clopen congruences on \mathbf{X} such that*

- (i) *the ordered set $\langle \mathcal{S}; \supseteq \rangle$ is directed, and*
- (ii) *for all $r \in R \cup \{=\}$ and $(a_1, \dots, a_m) \in X^m \setminus r^{\mathbf{X}}$, there is a congruence $\theta \in \mathcal{S}$ with $(a_1/\theta, \dots, a_m/\theta) \notin r^{\mathbf{X}/\theta}$.*

Then \mathbf{X} is isomorphic to a surjective inverse limit of finite quotients of \mathbf{X} , indexed over $\langle \mathcal{S}; \supseteq \rangle$, and so \mathbf{X} is profinite. Moreover, for every profinite topological structure \mathbf{X} , there is a set \mathcal{S} of clopen congruences on \mathbf{X} such that (i) and (ii) hold.

Proof. First assume that \mathbf{X} is compact, and let \mathcal{S} be a set of clopen congruences on \mathbf{X} such that (i) and (ii) hold. We can set up an inverse system on $\{\mathbf{X}/\theta \mid \theta \in \mathcal{S}\}$, where \mathcal{S} is ordered by reverse inclusion: for all $\theta \subseteq \lambda$ in \mathcal{S} , define the connecting homomorphism $\varphi_{\theta\lambda} : \mathbf{X}/\theta \rightarrow \mathbf{X}/\lambda$ by $\varphi_{\theta\lambda}(x/\theta) := x/\lambda$. Since \mathbf{X} is compact, each quotient \mathbf{X}/θ is finite. It is straightforward to prove that we can define an isomorphism $\psi : \mathbf{X} \rightarrow \varprojlim \{\mathbf{X}/\theta \mid \theta \in \mathcal{S}\}$ by $\psi(x)(\theta) := x/\theta$. (To prove that ψ is surjective, use the compactness of \mathbf{X} to find a member of $\bigcap \{z(\theta) \mid \theta \in \mathcal{S}\}$, for each $z \in \varprojlim \{\mathbf{X}/\theta \mid \theta \in \mathcal{S}\}$.) This inverse limit is clearly surjective.

Now assume that \mathbf{X} is profinite. We can assume that \mathbf{X} is equal to an inverse limit $\varprojlim \{\mathbf{X}_i \mid i \in I\}$ of finite structures. Define \mathcal{S} to be the set of all clopen congruences on \mathbf{X} . Then \mathcal{S} is directed under reverse inclusion, as it is closed under pairwise intersection.

To prove that (ii) holds, let $r \in R \cup \{=\}$ and $(a_1, \dots, a_m) \in X^m \setminus r^{\mathbf{X}}$. We must have $(a_1(i), \dots, a_m(i)) \notin r^{\mathbf{X}_i}$, for some $i \in I$. Let $\pi_i : \mathbf{X} \rightarrow \mathbf{X}_i$ denote the i th projection. There is a natural homomorphism $\mu_i : \mathbf{X}/\ker(\pi_i) \rightarrow \mathbf{X}_i$, given by $\mu_i(x/\ker(\pi_i)) := x(i)$. Now we have

$$(\mu_i(a_1/\ker(\pi_i)), \dots, \mu_i(a_m/\ker(\pi_i))) \notin r^{\mathbf{X}_i},$$

and so $(a_1/\ker(\pi_i), \dots, a_m/\ker(\pi_i)) \notin r^{\mathbf{X}/\ker(\pi_i)}$, as μ_i preserves r . \square

We now come to the first example of the importance of inverse limits to this study: as a corollary to the next lemma, we will see that the Boolean core of a uH-class \mathcal{M} is precisely the class of inverse limits of finite members of \mathcal{M} .

The operator $\mathbb{P}_{\text{fin}}^+$ denotes products over finite non-empty index sets (with the product topology).

Lemma 2.3. *Let \mathcal{K} be a class of compact Hausdorff topological structures. Then we have $\mathbb{IS}_c\mathbb{P}^+(\mathcal{K}) = \text{Inv}(\mathbb{IS}_c\mathbb{P}_{\text{fin}}^+(\mathcal{K}))$.*

Proof. It is easy to check that each inverse limit of members of $\mathbb{IS}_c\mathbb{P}_{\text{fin}}^+(\mathcal{K})$ is a topologically closed substructure of a product of members of \mathcal{K} . For the converse,

let \mathbf{X} be a topologically closed substructure of a product $\prod\{\mathbf{X}_i \mid i \in I\}$ of members of \mathcal{K} . We want to show that $\mathbf{X} \in \text{Inv}(\mathbb{I}\mathbb{S}_c\mathbb{P}_{\text{fin}}^+(\mathcal{K}))$.

Let \mathcal{S} be the set of all finite non-empty subsets of I , ordered by inclusion. For each $J \in \mathcal{S}$, let $- \upharpoonright_J : \mathbf{X} \rightarrow \prod\{\mathbf{X}_j \mid j \in J\}$ be the projection morphism. For all $K \supseteq J$ in \mathcal{S} , let $\varphi_{KJ} : \mathbf{X} \upharpoonright_K \rightarrow \mathbf{X} \upharpoonright_J$ be the projection. Then $\{\mathbf{X} \upharpoonright_J \mid J \in \mathcal{S}\}$ forms an inverse system of topological structures from $\mathbb{I}\mathbb{S}_c\mathbb{P}_{\text{fin}}^+(\mathcal{K})$. Define the inverse limit $\mathbf{Y} := \varprojlim\{\mathbf{X} \upharpoonright_J \mid J \in \mathcal{S}\}$.

We want to define an isomorphism $\psi : \mathbf{X} \rightarrow \mathbf{Y}$ by $\psi(x)(J) := x \upharpoonright_J$. For each $x \in X$, we have $\psi(x) \in Y$, since $\varphi_{KJ}(\psi(x)(K)) = \varphi_{KJ}(x \upharpoonright_K) = x \upharpoonright_J = \psi(x)(J)$, for all $K \supseteq J$ in \mathcal{S} . The map ψ is a homomorphism, since the projections are homomorphisms.

To see that ψ is injective and strong, let $r \in R \cup \{=\}$ and $(a_1, \dots, a_m) \in X^m$ such that $(\psi(a_1), \dots, \psi(a_m)) \in r^{\mathbf{Y}}$. For each $i \in I$, we have

$$\begin{aligned} (a_1(i), \dots, a_m(i)) &= (a_1 \upharpoonright_{\{i\}}(i), \dots, a_m \upharpoonright_{\{i\}}(i)) \\ &= (\psi(a_1)(\{i\})(i), \dots, \psi(a_m)(\{i\})(i)) \in r^{\mathbf{X}_i}. \end{aligned}$$

So $(a_1, \dots, a_m) \in r^{\mathbf{X}}$, as required. To see that ψ is continuous, consider a subbasic open set $U := \{y \in Y \mid y(J)(j) \in V\}$ of \mathbf{Y} , where $j \in J \in \mathcal{S}$ and V is open in \mathbf{X}_j . We have $\psi^{-1}(U) = \{x \in X \mid x(j) \in V\}$, which is open in \mathbf{X} .

It remains to verify that ψ is surjective. Let $y \in Y$. For each $J \in \mathcal{S}$, define the closed set $V_J := \{x \in X \mid x \upharpoonright_J = y(J)\}$ in \mathbf{X} . Since $y \in Y$, we know that $y(J) \in X \upharpoonright_J$ and therefore $V_J \neq \emptyset$, for each $J \in \mathcal{S}$. The set $\{V_J \mid J \in \mathcal{S}\}$ is closed under finite intersection: $V_{J_1} \cap \dots \cap V_{J_n} = V_{J_1 \cup \dots \cup J_n} \neq \emptyset$. So, as \mathbf{X} is compact, we can choose $x \in \bigcap\{V_J \mid J \in \mathcal{S}\}$. For each $J \in \mathcal{S}$, we have $\psi(x)(J) = x \upharpoonright_J = y(J)$. Thus $\psi(x) = y$. \square

Corollary 2.4.

- (i) For each class \mathcal{K} of finite structures, we have $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{K}) = \text{Inv}([\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathcal{K})]_{\text{fin}})$.
- (ii) For each uH-class \mathcal{M} , we have $\mathbf{M}_{\text{Bc}} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}}) = \text{Inv}(\mathcal{M}_{\text{fin}})$.

In the case that \mathcal{K} is taken to be a small finite set of finite structures, we see that part (i) of this result offers us a clear tradeoff. A structure in the topological prevariety generated by \mathcal{K} can be represented either as an arbitrary closed subdirect product with a few well-understood factors, or as a well-understood subdirect product with arbitrarily large finite factors. This tradeoff becomes hidden in the statement of part (ii).

Remark 2.5. The original notion of the standardness of a finite structure [10] allows partial operations in its type (partial operations being a provably essential feature of many natural dualities [8]). However, a number of complications arise in the present context if partial operations are allowed. To begin with, we need a notion of congruence. But under the natural choices, there may no longer be a maximal congruence contained in a given equivalence. Hence syntactic congruences are not necessarily available. These problems culminate in the failure of Corollary 2.4. There exists a finite partial structure \mathbf{M} and $\mathbf{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ such that $\mathbf{X} \notin \text{Inv}([\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})]_{\text{fin}})$; see Davey [17]. We have no general way of knowing when the methods of this paper apply to partial structures and when they do not. Accordingly, we have omitted partial structures from our considerations here.

The following theorem, which is an immediate consequence of Corollary 2.4 and the Preservation Theorem 1.2, provides an overview of the component subsidiary conditions that constitute standardness.

Theorem 2.6. *Let \mathcal{M} be a uH-class. Then*

$$\mathcal{M}_{\text{Bc}} := \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}}) = \text{Inv}(\mathcal{M}_{\text{fin}}) \left\{ \begin{array}{c} \subseteq \\ \subseteq \end{array} \begin{array}{c} \mathcal{M}_{\text{pro}} \\ [\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}} \end{array} \begin{array}{c} \subseteq \\ \subseteq \end{array} \right\} \mathcal{M}_{\text{Bt}},$$

and \mathcal{M} is standard if and only if all of these classes coincide. In particular, if \mathcal{M} is standard, then every member of \mathcal{M}_{Bt} is profinite.

We can establish that \mathcal{M} is standard by verifying that either the top two inclusions or the bottom two inclusions are equalities. We can establish that \mathcal{M} is non-standard by verifying that any one of these inclusions is proper.

We shall consider each of the four inclusions one at a time, starting with the top-left inclusion: $\text{Inv}(\mathcal{M}_{\text{fin}}) \subseteq \mathcal{M}_{\text{pro}}$. We first show that there can be a profinite structure in \mathcal{M}_{Bt} that is not an inverse limit of finite members of \mathcal{M} itself.

Example 2.7. *Let \mathcal{A} denote the uH-class of torsion-free abelian groups. Then we have $\text{Inv}(\mathcal{A}_{\text{fin}}) \neq \mathcal{A}_{\text{pro}}$, and therefore \mathcal{A} is non-standard.*

Proof. We write abelian groups additively, so that \mathcal{A} is defined within abelian groups by the set of uH-formulae $\{nx \approx 0 \Rightarrow x \approx 0 \mid n \in \mathbb{N}\}$. Let $\mathbf{1}$ denote the one-element group. Since $\mathcal{A}_{\text{fin}} = \mathbb{I}(\mathbf{1})$, we have $\text{Inv}(\mathcal{A}_{\text{fin}}) = \mathbb{I}(\mathbf{1})$.

For each $k \in \mathbb{N}$, let \mathbf{Z}_k denote the group of integers modulo k . For $n \in \mathbb{N}$, we define the connecting homomorphism $\varphi_n : \mathbf{Z}_{2^{n+1}} \rightarrow \mathbf{Z}_{2^n}$ by $\varphi_n(a) := a \pmod{2^n}$. Now define $\mathbf{X} := \varprojlim \{\mathbf{Z}_{2^n} \mid n \in \mathbb{N}\}$. For every $x \in X \setminus \{0\}$, the order of each non-zero coordinate of x is twice the order of the previous coordinate, and thus x has infinite order. Hence $\mathbf{X} \in \mathcal{A}_{\text{pro}} \setminus \text{Inv}(\mathcal{A}_{\text{fin}})$. \square

The crux of the previous example is that, while the factors of the inverse system are all quotients of the torsion-free inverse limit, the class of torsion-free abelian groups is not closed under taking quotients. In general, we can always express a profinite structure as an inverse limit of finite quotients, by Lemma 2.2. So, using Theorem 2.6, we obtain the following result.

Lemma 2.8. *Let \mathcal{M} be a uH-class. The following conditions are ordered from strongest to weakest:*

- (i) \mathcal{M} is an atomic class;
- (ii) \mathcal{M} is closed under finite quotients;
- (iii) $\text{Inv}(\mathcal{M}_{\text{fin}}) = \mathcal{M}_{\text{pro}}$.

We now turn to the top-right inclusion of Theorem 2.6: $\mathcal{M}_{\text{pro}} \subseteq \mathcal{M}_{\text{Bt}}$. The following is a variant of [9, 7.3].

Lemma 2.9. *Let \mathcal{M} be a uH-class. The following conditions are ordered from strongest to weakest:*

- (i) \mathcal{M} is a class of algebras and contains an infinite subdirectly irreducible algebra that admits a compatible Boolean topology;
- (ii) \mathcal{M} contains a structure that admits a compatible Boolean topology but is not residually finite;
- (iii) $\mathcal{M}_{\text{pro}} \neq \mathcal{M}_{\text{Bt}}$;
- (iv) \mathcal{M} is non-standard.

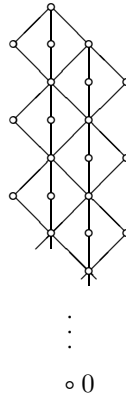


FIGURE 1. A non-profinite Boolean topological lattice.

Example 2.10. Let \mathcal{L} denote the uH-class of all lattices. Then $\mathcal{L}_{\text{pro}} \neq \mathcal{L}_{\text{Bt}}$, and so \mathcal{L} is non-standard.

Proof. Let \mathbf{L} be the lattice of Figure 1. Then \mathbf{L} is subdirectly irreducible, and \mathbf{L} admits the one-point compactification topology, with 0 the compactification point and $L \setminus \{0\}$ discrete (Clinkenbeard [13]). Now use the previous lemma. \square

Lemma 2.11. Let \mathfrak{X} be a Boolean topological structure. If there is a finite set of terms that determines syntactic congruences on \mathbf{X} , then \mathfrak{X} is profinite.

Proof. Assume that \mathfrak{X} has FDSC, and let \mathcal{S} denote the set of all clopen congruences on \mathfrak{X} . Then \mathcal{S} is directed under reverse inclusion. To apply Lemma 2.2, we let $r \in R \cup \{=\}$ and $(a_1, \dots, a_m) \in X^m \setminus r^{\mathbf{X}}$. Since $r^{\mathbf{X}}$ is closed, there is an open subset U of \mathfrak{X}^m with $(a_1, \dots, a_m) \in U$ and $U \cap r^{\mathbf{X}} = \emptyset$. We can choose U to be a clopen basis set, and so $U = U_1 \times \dots \times U_m$, for clopen subsets U_i of \mathfrak{X} . Define the clopen equivalence θ on \mathfrak{X} by $(x, y) \in \theta$ if

$$x \in U_i \iff y \in U_i, \text{ for all } i = 1, \dots, m.$$

By Lemma 1.5, we have $\text{Syn}(\theta) \in \mathcal{S}$. We also have $(a_1/\text{Syn}(\theta), \dots, a_m/\text{Syn}(\theta)) \notin r^{\mathbf{X}/\text{Syn}(\theta)}$. By Lemma 2.2, we conclude that \mathfrak{X} is profinite. \square

Corollary 2.12. Let \mathcal{M} be a uH-class. If \mathcal{M} has FDSC, then $\mathcal{M}_{\text{pro}} = \mathcal{M}_{\text{Bt}}$.

Note that every class of purely relational structures has FDSC: syntactic congruences are trivially determined by $F := \{x\}$. Combining Theorem 2.6, Lemma 2.8 and Corollary 2.12 immediately gives us the following generalization of the FDSC-HSP Theorem 4.3 of [9].

Theorem 2.13. If a uH-class is closed under finite quotients and has FDSC, then it is standard.

Every variety of algebras is closed under quotients, and many familiar varieties of algebras have FDSC [9]: the variety generated by a finite lattice or a finite unary algebra, and every variety of groups, rings or semigroups. Wang [37] proved that every finitely generated congruence-distributive variety of algebras has a finiteness

property that is equivalent to FDSC [9, 6.9]. So every congruence-distributive variety $\mathcal{V} := \text{Var}(\mathbf{M})$, generated by a finite algebra \mathbf{M} , is standard. We know that $\mathcal{V} = \mathbb{ISP}(\mathcal{K})$, for some finite set \mathcal{K} of finite subdirectly irreducible algebras. It now follows that we also have $\mathcal{V}_{\text{Bt}} = \mathbb{IS}_c\mathbb{P}(\mathcal{K})$.

Occasionally, we find a uH-class that is closed under quotients although it is not an atomic class; for example, the class $\mathcal{M} := \text{Mod}(x \sim y \Rightarrow y \sim x)$ of all graphs. Since the class \mathcal{M} is purely relational, it has FDSC and is therefore standard.

Example 2.14. *The uH-class of graphs is standard.*

Remark 2.15. We shall present new general conditions for standardness in Section 9. For a structure \mathbf{M} with only finitely many relations in its type, the uH-class $\mathbb{ISP}^+(\mathbf{M})$ is standard if $\mathbb{HSP}(\mathbf{M})$ has FDSC and \mathbf{M} is injective in $\mathbb{HSP}(\mathbf{M})$; see Theorem 9.2.

Now consider the bottom-left inclusion of Theorem 2.6: $\mathcal{M}_{\text{Bc}} \subseteq [\mathbb{ISP}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}}$. Equality of this inclusion is what we have called pre-standardness. The following example illustrates a failure of standardness resulting from a failure of pre-standardness. A **uniar** is a unary algebra with only one operation.

Example 2.16. *The uH-class of all unars is not even pre-standard.*

Proof. Let \mathcal{M} denote the uH-class of all unars. Define the Boolean topological uniar $\mathbf{X} := \langle \mathbb{Z} \cup \{\infty\}; s, \mathcal{T} \rangle$ such that

- s is the successor function on the set \mathbb{Z} of integers, and s fixes ∞ ,
- \mathcal{T} is the one-point compactification topology on $\mathbb{Z} \cup \{\infty\}$, with ∞ the compactification point and \mathbb{Z} discrete.

To see that $\mathbf{X} \in [\mathbb{ISP}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}}$, let $j, k \in \mathbb{Z}$ with $j \not\equiv k \pmod{n}$, for some $n \in \mathbb{N}$. Define the finite uniar $\mathbf{U}_n := \langle \{0, 1, \dots, n-1\} \cup \{\infty\}; s \rangle$, where $s(i) := i \oplus_n 1$, for $i < n$, and $s(\infty) := \infty$. Then there is a homomorphism from \mathbf{X} onto \mathbf{U}_n that separates $\{j, k, \infty\}$, and it follows that $\mathbf{X} \in [\mathbb{ISP}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}}$. However, every continuous homomorphism from \mathbf{X} into a finite uniar is constant. By the Separation Theorem 1.3, we conclude that $\mathbf{X} \notin \mathcal{M}_{\text{Bc}}$. \square

Finally, we address the bottom-right inclusion: $[\mathbb{ISP}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}} \subseteq \mathcal{M}_{\text{Bt}}$. This inclusion is proper for the class of torsion-free abelian groups, for instance; see 2.7. A simple condition for equality of this inclusion is that \mathcal{M} is generated by its finite members, in the sense that $\mathcal{M} = \mathbb{ISP}^+(\mathcal{M}_{\text{fin}})$. This condition holds for many familiar uH-classes; in particular, for each uH-class $\mathbb{ISP}^+(\mathbf{M})$ whose Boolean core $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ is the dual category under a natural duality. We can show that many of our definability conditions are equivalent for a uH-class $\mathcal{M} = \mathbb{ISP}^+(\mathcal{M}_{\text{fin}})$.

Proposition 2.17. *Let \mathcal{M} be a uH-class with Boolean core $\mathcal{M}_{\text{Bc}} := \mathbb{IS}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}})$. Assume that $\mathcal{M} = \mathbb{ISP}^+(\mathcal{M}_{\text{fin}})$. Then the following are equivalent:*

- (i) \mathcal{M} is standard;
- (ii) \mathcal{M} is pre-standard;
- (iii) \mathcal{M}_{Bc} is Horn definable;
- (iv) \mathcal{M}_{Bc} is universally definable.

Proof. The implications (i) \Rightarrow (ii) and (i) \Rightarrow (iii) \Rightarrow (iv) are trivial. If \mathcal{M} is pre-standard, then $\mathcal{M}_{\text{Bc}} = [\mathbb{ISP}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}} = \mathcal{M}_{\text{Bt}}$, and so \mathcal{M} is standard. Thus (ii) \Rightarrow (i). Finally, we prove (iv) \Rightarrow (i).

Assume that $\mathcal{M}_{\text{Bc}} = \text{Mod}_{\text{Bt}}(\Sigma)$, with Σ a set of universal sentences, and let $\mathfrak{X} \in \mathcal{M}_{\text{Bt}}$. Then $\mathbf{X} \in \mathcal{M} = \mathbb{ISP}^+(\mathcal{M}_{\text{fin}})$, and therefore $\mathbf{X} \cong \mathbf{Y} \leq \prod\{\mathbf{M}_i \mid i \in I\}$, with each \mathbf{M}_i from \mathcal{M}_{fin} . Define $\mathfrak{Z} := \prod\{\mathbf{M}_i \mid i \in I\}$, where each \mathbf{M}_i has the discrete topology and \mathfrak{Z} has the product topology. Then $\mathfrak{Z} \in \mathcal{M}_{\text{Bc}}$, and so $\mathbf{Z} \models \Sigma$. Since Σ is universal, we conclude that $\mathbf{X} \models \Sigma$. Thus $\mathfrak{X} \in \text{Mod}_{\text{Bt}}(\Sigma) = \mathcal{M}_{\text{Bc}}$. \square

The definability condition missing from the previous result is:

(v) \mathcal{M}_{Bc} is first-order definable.

In Section 4, we exhibit a finite lattice \mathbf{M}_{3-3} such that $\mathcal{M} := \mathbb{ISP}^+(\mathbf{M}_{3-3})$ satisfies condition (v) but fails conditions (i) through (iv). Furthermore, the previous result does not hold in general for $\mathcal{M} \neq \mathbb{ISP}^+(\mathcal{M}_{\text{fin}})$. Consider again the uH-class \mathcal{A} of torsion-free abelian groups. Then \mathcal{A}_{Bc} consists of all 1-element groups, and so \mathcal{A} satisfies conditions (ii) through (v). But \mathcal{A} is not standard, by Example 2.7.

The previous result suggests the long-term project of classifying finite structures according to the axiomatizability of the topological quasivarieties that they generate.

Corollary 2.18. *Let \mathbf{M} be a finite structure and define $\mathcal{M} := \mathbb{ISP}^+(\mathbf{M})$. Then the topological prevariety $\mathcal{M}_{\text{Bc}} = \mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$ satisfies exactly one of the following:*

- (i) $\mathcal{M}_{\text{Bc}} = \text{Mod}_{\text{Bt}}(\text{Th}_{\text{uH}}(\mathbf{M}))$, that is, the structure \mathbf{M} is standard;
- (ii) $\mathcal{M}_{\text{Bc}} = \text{Mod}_{\text{Bt}}(\Sigma)$, for some set Σ of first-order sentences but for no set Σ of universal sentences;
- (iii) \mathcal{M}_{Bc} is not first-order definable.

In this paper, we will develop techniques to help classify finite structures into one of these three types.

The results of this section yield a fundamental description of Boolean topological structures. The following theorem is a simple generalization of Theorem 8.1 of [9], which compiled a collection of previously known results for topological *algebras*. A topological structure \mathfrak{X} is said to be **topologically residually finite** if, for every $r \in R \cup \{=\}$ and $(a_1, \dots, a_m) \in X^m \setminus r^{\mathbf{X}}$, there is a continuous homomorphism φ from \mathfrak{X} to a finite structure \mathbf{Z} such that $(\varphi(a_1), \dots, \varphi(a_m)) \notin r^{\mathbf{Z}}$.

Theorem 2.19. *Let \mathfrak{X} be a compact topological structure. Then the following conditions are equivalent (and imply that the topology on \mathfrak{X} is Boolean):*

- (i) \mathfrak{X} is topologically residually finite;
- (ii) \mathfrak{X} is profinite;
- (iii) \mathfrak{X} is a closed subdirect product of finite discrete structures.

Moreover, if there is a finite set of terms that determines syntactic congruences on \mathbf{X} , then these conditions are all equivalent to the topology on \mathfrak{X} being Boolean.

Proof. Conditions (i) and (iii) are equivalent, by the Separation Theorem 1.3. Conditions (ii) and (iii) are equivalent, using Corollary 2.4. Condition (iii) implies that the topology on \mathfrak{X} is Boolean, by Tychonoff's Product Theorem. Finally, if \mathfrak{X} has FDSC and a Boolean topology, then condition (ii) holds, by Lemma 2.11. \square

3. INVERSE LIMIT TECHNIQUES

The previous section suggests where to look for a structure $\mathfrak{X} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$ to witness that a uH-class \mathcal{M} is non-standard. If our uH-class \mathcal{M} has FDSC but is non-standard, then Theorem 2.6 and Corollary 2.12 tell us that there exists an

inverse limit $\underline{\mathbf{X}}$ of finite structures that is in \mathcal{M}_{Bt} but not in $\text{Inv}(\mathcal{M}_{\text{fin}})$. In this section, we will present several techniques for constructing such an inverse limit witnessing non-standardness. In each case, the constructed inverse limit will also witness that \mathcal{M}_{Bc} is not even universally definable. One technique will also allow us to conclude that \mathcal{M} is not finitely axiomatizable.

Throughout this section, we will be working with an inverse system of finite structures $\{\mathbf{X}_n \mid n \in \mathbb{N}\}$, indexed over the positive integers, with the connecting homomorphisms $\{\varphi_{mn} \mid m \geq n \text{ in } \mathbb{N}\}$. For convenience, define

$$\varphi_n := \varphi_{(n+1)n} : \mathbf{X}_{n+1} \rightarrow \mathbf{X}_n,$$

for each $n \in \mathbb{N}$. Then the connecting maps of the inverse system are determined by the maps in $\{\varphi_n \mid n \in \mathbb{N}\}$.

We seek conditions under which the inverse limit $\underline{\mathbf{X}} := \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ is in \mathcal{M}_{Bt} but not \mathcal{M}_{Bc} . To help establish that $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}}$, we use the following test for whether the inverse limit $\underline{\mathbf{X}}$ satisfies a given universal sentence. The ultraproduct construction plays an important role in the proof of this result, and also in Section 5. We refer to Burris and Sankappanavar [6, V.2] for a nice review of ultraproducts.

Lemma 3.1. *Let $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ be an inverse limit of finite structures, let $\alpha(x_1, \dots, x_k)$ be a quantifier-free formula and let $a_1, \dots, a_k \in X$. The following are equivalent:*

- (i) $\mathbf{X} \models \alpha(a_1, \dots, a_k)$;
- (ii) the set $W_\alpha := \{n \in \mathbb{N} \mid \mathbf{X}_n \models \alpha(a_1(n), \dots, a_k(n))\}$ is cofinite in \mathbb{N} .

Proof. Let \mathcal{U} be any non-principal ultrafilter on \mathbb{N} , and define the ultraproduct $\mathbf{Z} := \prod_{\mathcal{U}} \{\mathbf{X}_n \mid n \in \mathbb{N}\}$. Let $\eta : \mathbf{X} \rightarrow \mathbf{Z}$ denote the restriction of the natural homomorphism from $\prod \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ to \mathbf{Z} . We will prove that η is an embedding.

Let $r \in R \cup \{=\}$ and let $b_1, \dots, b_m \in X$ with $(\eta(b_1), \dots, \eta(b_m)) \in r^{\mathbf{Z}}$. By the definition of ultraproduct, the set

$$W := \{n \in \mathbb{N} \mid (b_1(n), \dots, b_m(n)) \in r^{\mathbf{X}_n}\}$$

is in \mathcal{U} . Since \mathcal{U} is non-principal, the set W is infinite. As all the connecting maps $\{\varphi_n \mid n \in \mathbb{N}\}$ preserve r , we must have $W = \mathbb{N}$. So $(b_1, \dots, b_m) \in r^{\mathbf{X}}$, as required.

We have shown that \mathbf{X} is isomorphic to a substructure of \mathbf{Z} , via the embedding η . Since $\alpha(a_1, \dots, a_k)$ is variable-free statement about \mathbf{X} , it follows that $\mathbf{X} \models \alpha(a_1, \dots, a_k)$ if and only if $\mathbf{Z} \models \alpha(\eta(a_1), \dots, \eta(a_k))$. By Łoś's Ultraproduct Theorem, we now have $\mathbf{X} \models \alpha(a_1, \dots, a_k)$ if and only if $W_\alpha \in \mathcal{U}$.

It remains to prove that $W_\alpha \in \mathcal{U}$ if and only if W_α is cofinite. We first show that W_α is either finite or cofinite. For an atomic formula $\beta(x_1, \dots, x_k)$, the subset W_β of \mathbb{N} is decreasing (and therefore finite or cofinite), since the connecting maps $\{\varphi_n \mid n \in \mathbb{N}\}$ are homomorphisms. For quantifier-free formulæ $\gamma(x_1, \dots, x_k)$ and $\delta(x_1, \dots, x_k)$, we have

$$W_{\neg\gamma} = \mathbb{N} \setminus W_\gamma, \quad W_{\gamma \vee \delta} = W_\gamma \cup W_\delta \quad \text{and} \quad W_{\gamma \wedge \delta} = W_\gamma \cap W_\delta.$$

By induction, the set W_α is either finite or cofinite. Since \mathcal{U} is non-principal, it now follows that $W_\alpha \in \mathcal{U}$ if and only if W_α is cofinite. \square

We want to have a ‘non-topological’ method for showing that an inverse limit $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ does not belong to \mathcal{M}_{Bc} . So we will not use the Separation Theorem 1.3 directly. Instead, we say that $\underline{\mathbf{X}}$ is **pointwise non-separable** with respect to \mathcal{M} if the following condition holds:

there exist $r \in R \cup \{=\}$ and $(b_1, \dots, b_m) \in X^m \setminus r^{\mathbf{X}}$ such that, for each $n \in \mathbb{N}$, each finite structure $\mathbf{M} \in \mathcal{M}_{\text{fin}}$ and each homomorphism $\psi : \mathbf{X}_n \rightarrow \mathbf{M}$, we have $(\psi(b_1(n)), \dots, \psi(b_m(n))) \in r^{\mathbf{M}}$.

It is easy to use the Separation Theorem 1.3 to check that, if $\underline{\mathbf{X}} \notin \mathcal{M}_{\text{Bc}}$ with $\underline{\mathbf{X}} \cong \mathbf{1}$, then this condition holds.

For the above condition to imply that $\underline{\mathbf{X}} \notin \mathcal{M}_{\text{Bc}}$, we need to know that certain morphisms out of the inverse limit $\underline{\mathbf{X}}$ factor through a projection. The following lemma gives us assumptions that guarantee that this factoring occurs. In particular, we assume that the set R of relation symbols is finite. (Nevertheless, all the results in this section can be adapted to fit the case where R is infinite; see Remark 3.8.) The proof of the lemma, which involves a tricky compactness argument, can be adapted from the proof of Lemma 2.2.6 in [8].

Lemma 3.2. *Assume that R is finite. Let $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ be a surjective inverse limit of finite structures and let $\varphi : \underline{\mathbf{X}} \rightarrow \mathbf{M}$ be a morphism, with \mathbf{M} finite. Then there exist $n \in \mathbb{N}$ and a morphism $\psi : \mathbf{X}_n \rightarrow \mathbf{M}$ such that $\varphi = \psi \circ \pi_n$.*

Lemma 3.3. *Assume that R is finite. Let $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ be a surjective inverse limit of finite structures, and let \mathcal{M} be a uH-class. If $\underline{\mathbf{X}}$ is pointwise non-separable with respect to \mathcal{M} , then $\underline{\mathbf{X}} \notin \mathcal{M}_{\text{Bc}}$.*

Proof. Assume that $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bc}}$. To see that pointwise non-separability fails, choose any $r \in R \cup \{=\}$ and $(b_1, \dots, b_m) \in X^m \setminus r^{\mathbf{X}}$. By the Separation Theorem 1.3, there are $\mathbf{M} \in \mathcal{M}_{\text{fin}}$ and a morphism $\varphi : \underline{\mathbf{X}} \rightarrow \mathbf{M}$ such that $(\varphi(b_1), \dots, \varphi(b_m)) \notin r^{\mathbf{M}}$. By the previous lemma, there exist $n \in \mathbb{N}$ and a homomorphism $\psi : \mathbf{X}_n \rightarrow \mathbf{M}$ such that $\varphi = \psi \circ \pi_n$, and so we have

$$(\psi(b_1(n)), \dots, \psi(b_m(n))) = (\varphi(b_1), \dots, \varphi(b_m)) \notin r^{\mathbf{M}}.$$

Thus $\underline{\mathbf{X}}$ is not pointwise non-separable with respect to \mathcal{M} . \square

Lemma 3.4. *Assume that R is finite. Let $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ be a surjective inverse limit of finite structures, and let \mathcal{M} be a uH-class. Assume that*

- (i) $\underline{\mathbf{X}}$ is pointwise non-separable with respect to \mathcal{M} , and
- (ii) $\mathbf{X} \models \text{Th}_{\forall}(\mathcal{M}_{\text{fin}})$.

Then $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$ and \mathcal{M}_{Bc} is not universally definable.

Proof. By Lemma 3.3, the pointwise non-separability of $\underline{\mathbf{X}}$ implies that $\underline{\mathbf{X}} \notin \mathcal{M}_{\text{Bc}}$. Since $\mathbf{X} \models \text{Th}_{\forall}(\mathcal{M}_{\text{fin}})$, we have $\mathbf{X} \models \text{Th}_{\text{uH}}(\mathcal{M})$ and therefore $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}}$. Also, since $\mathcal{M}_{\text{fin}} \subseteq \mathcal{M}_{\text{Bc}}$, we have $\mathbf{X} \models \text{Th}_{\forall}(\mathcal{M}_{\text{Bc}})$. As $\underline{\mathbf{X}} \notin \mathcal{M}_{\text{Bc}}$, we can conclude that \mathcal{M}_{Bc} is not universally definable. \square

Each of the three techniques for proving non-standardness that we present here will use a different condition for showing that $\mathbf{X} \models \text{Th}_{\forall}(\mathcal{M}_{\text{fin}})$.

First Inverse Limit Technique 3.5. *Assume that the set R of relation symbols is finite. Let $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ be a surjective inverse limit of finite structures, and let \mathcal{M} be a uH-class. Assume that $\underline{\mathbf{X}}$ is pointwise non-separable with respect to \mathcal{M} and that the following condition holds.*

$F(\underline{\mathbf{X}}, \mathcal{M})$: *For all $n \geq 2$ and each subset Y of X_n with at most n elements, if the map $\varphi_{n-1} : \mathbf{X}_n \rightarrow \mathbf{X}_{n-1}$ is one-to-one on Y , then $\text{Sg}_{\mathbf{X}_n}(Y) \in \mathcal{M}$.*

Then $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$ and \mathcal{M}_{Bc} is not universally definable, both of which imply that \mathcal{M} is non-standard.

Proof. We use Lemma 3.4. Condition (i) of 3.4 holds by assumption. To verify (ii), let $\alpha(x_1, \dots, x_k) \in \text{Th}_{\forall}(\mathcal{M}_{\text{fin}})$ and let $a_1, \dots, a_k \in X$. Let $\ell := |\{a_1, \dots, a_k\}| \leq k$. Since $\underline{\mathbf{X}}$ is the inverse limit, there is an $m \in \mathbb{N}$ such that, for all $n \geq m$, we have $|\{a_1(n), \dots, a_k(n)\}| = \ell$. We can assume that $m \geq k$.

Now let $n > m$. Define $Y := \{a_1(n), \dots, a_k(n)\} \subseteq X_n$. Then $|Y| = \ell \leq n$ and φ_{n-1} is one-to-one on Y . So we can conclude from $F(\underline{\mathbf{X}}, \mathcal{M})$ that $\text{Sg}_{\mathbf{X}_n}(Y) \in \mathcal{M}_{\text{fin}}$. Thus $\text{Sg}_{\mathbf{X}_n}(Y) \models \alpha(a_1(n), \dots, a_k(n))$ and therefore $\mathbf{X}_n \models \alpha(a_1(n), \dots, a_k(n))$. It now follows by Lemma 3.1 that $\mathbf{X} \models \alpha(a_1, \dots, a_k)$. Hence (ii) of 3.4 holds. \square

Remark 3.6. In some cases, we can further restrict the condition $F(\underline{\mathbf{X}}, \mathcal{M})$ of the FILT 3.5. If there is a basis Σ for $\text{Th}_{\forall}(\mathcal{M}_{\text{fin}})$ such that Σ uses only k variables, then we can restrict $F(\underline{\mathbf{X}}, \mathcal{M})$ to those Y with at most k elements. The proof is exactly the same, except that we consider only $\alpha(x_1, \dots, x_k) \in \Sigma$.

Remark 3.7. Sometimes one inverse limit $\underline{\mathbf{X}}$ will be able to witness that every uH-class in an interval of uH-classes is non-standard. Let \mathcal{N} and \mathcal{P} be uH-classes with $\mathcal{N} \subseteq \mathcal{P}$. If we can show that $\underline{\mathbf{X}}$ is pointwise non-separable with respect to \mathcal{P} and that the condition $F(\underline{\mathbf{X}}, \mathcal{N})$ holds, then every uH-class \mathcal{M} such that $\mathcal{N} \subseteq \mathcal{M} \subseteq \mathcal{P}$ is non-standard. The remaining two inverse-limit techniques can be used similarly.

Remark 3.8. The FILT 3.5 (indeed, all three inverse-limit techniques) can be adapted for the case where R is infinite. We just need assumptions to make Lemma 3.2 work. The assumption that R is finite can be replaced by the assumption that every projection $\pi_n : \underline{\mathbf{X}} \rightarrow \mathbf{X}_n$ is strong. To guarantee this, it is enough to check that every connecting morphism $\varphi_n : \mathbf{X}_{n+1} \rightarrow \mathbf{X}_n$ is strong and surjective. (For the proof of this fact, adapt the proof of Lemma 2.1.)

By strengthening the condition $F(\underline{\mathbf{X}}, \mathcal{M})$ above, we obtain a new technique that can also be used to establish that the uH-class \mathcal{M} is not finitely axiomatizable.

Second Inverse Limit Technique 3.9. *Assume that the set R of relation symbols is finite. Let $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ be a surjective inverse limit of finite structures, and let \mathcal{M} be a uH-class. Assume that $\underline{\mathbf{X}}$ is pointwise non-separable with respect to \mathcal{M} and that the following condition holds.*

$S(\underline{\mathbf{X}}, \mathcal{M})$: *For all $n \in \mathbb{N}$, each substructure of \mathbf{X}_n that is generated by at most n elements belongs to \mathcal{M} .*

Then $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$ and \mathcal{M}_{Bc} is not universally definable, both of which imply that \mathcal{M} is non-standard. Furthermore, the uH-class \mathcal{M} is not finitely axiomatizable.

Proof. By the FILT 3.5, we know that $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$ and that \mathcal{M}_{Bc} is not universally definable. Now suppose that $\text{Th}_{\text{uH}}(\mathcal{M})$ has a finite basis. Taking the conjunction of the uH-formulae in the basis gives us a quantifier-free formula $\alpha(x_1, \dots, x_k)$

that axiomatizes \mathcal{M} . By $S(\underline{\mathbf{X}}, \mathcal{M})$, each substructure of \mathbf{X}_n that is generated by at most n elements satisfies $\alpha(x_1, \dots, x_k)$, for all $n \in \mathbb{N}$. So, for all $n \geq k$, we have $\mathbf{X}_n \models \alpha(x_1, \dots, x_k)$ and therefore $\mathbf{X}_n \in \mathcal{M}$. On the other hand, the pointwise non-separability of $\underline{\mathbf{X}}$ implies that $\mathbf{X}_n \notin \mathcal{M}$, for all sufficiently large $n \in \mathbb{N}$. \square

Third Inverse Limit Technique 3.10. *Assume that the set R of relation symbols is finite. Let $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ be a surjective inverse limit of finite structures, and let \mathcal{M} be a uH-class. Assume that $\underline{\mathbf{X}}$ is pointwise non-separable with respect to \mathcal{M} and that the following condition holds.*

$T(\underline{\mathbf{X}}, \mathcal{M})$: *There exist $Y_n \subseteq X_n$, for all $n \in \mathbb{N}$, such that $\varphi_{mn}(X_m \setminus Y_m) \subseteq Y_n$ and $\text{Sg}_{\mathbf{X}_n}(Y_n) \in \mathcal{M}$, for all $m > n$ in \mathbb{N} .*

Then $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$ and \mathcal{M}_{Bc} is not universally definable, both of which imply that \mathcal{M} is non-standard.

Proof. We use Lemma 3.4 again. We must verify that $\mathbf{X} \models \text{Th}_{\forall}(\mathcal{M}_{\text{fin}})$. So let $\alpha(x_1, \dots, x_k) \in \text{Th}_{\forall}(\mathcal{M}_{\text{fin}})$ and $a_1, \dots, a_k \in X$. First consider any $i \in \{1, 2, \dots, k\}$. If $a_i(m) \notin Y_m$, for some $m \in \mathbb{N}$, then $T(\underline{\mathbf{X}}, \mathcal{M})$ tells us that $a_i(n) \in Y_n$, for all $n \in \mathbb{N} \setminus \{m\}$. So there is at most one $n \in \mathbb{N}$ for which $a_i(n) \notin Y_n$.

We can choose $\ell \in \mathbb{N}$ large enough so that, for all $n \geq \ell$ and $i \in \{1, 2, \dots, k\}$, we have $a_i(n) \in Y_n$. Now, for all $n \geq \ell$, we have $\{a_1(n), \dots, a_k(n)\} \subseteq Y_n$ with $\text{Sg}_{\mathbf{X}_n}(Y_n) \in \mathcal{M}_{\text{fin}}$. By Lemma 3.1, we conclude that $\mathbf{X} \models \alpha(a_1, \dots, a_k)$. \square

We will now prove, under some quite general conditions, that any witness to non-standardness must be isomorphic to an inverse limit with the properties common to our three techniques. In particular, we place a condition on the topology of the witness to non-standardness, described in the following lemma.

Lemma 3.11. *Let $\underline{\mathbf{X}}$ be a Boolean topological space. The following are equivalent:*

- (i) $\underline{\mathbf{X}}$ has a countable basis of clopen sets;
- (ii) $\underline{\mathbf{X}}$ has a countable basis;
- (iii) $\underline{\mathbf{X}}$ has a countable number of clopen sets;
- (iv) $\underline{\mathbf{X}}$ is a metric space;
- (v) $\underline{\mathbf{X}}$ embeds into the Cantor space 2^ω , where 2 is the 2-element discrete space.

Proof. We sketch the proof. The implication (iii) \Rightarrow (i) is trivial. For (i) \Rightarrow (v), assume that \mathcal{B} is a countable basis for $\underline{\mathbf{X}}$ consisting of clopen sets. Then we can define the embedding $\varphi : \underline{\mathbf{X}} \rightarrow 2^\omega$ by $\varphi(x)(U) = 1$, if $x \in U$, and $\varphi(x)(U) = 0$, otherwise. The implication (v) \Rightarrow (iv) follows as the Cantor space is metric (it can be obtained by removing open middle thirds from the unit interval). The implication (iv) \Rightarrow (ii) comes from a standard result of topology (every compact metric space is second countable). The remaining implication, (ii) \Rightarrow (iii), is straightforward. \square

Proposition 3.12. *Let $\underline{\mathbf{X}}$ be a Boolean topological structure that has FDSC and has a countable basis of clopen sets. Then $\underline{\mathbf{X}}$ is isomorphic to a surjective inverse limit of finite structures, indexed over \mathbb{N} . Moreover, if $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$, for some uH-class \mathcal{M} , then this inverse limit is pointwise non-separable with respect to \mathcal{M} .*

Proof. We will apply Lemma 2.2. Let U_1, U_2, U_3, \dots be a countable basis for $\underline{\mathbf{X}}$, with each U_n clopen. We can construct clopen equivalences $\theta_1 \supseteq \theta_2 \supseteq \theta_3 \supseteq \dots$ on $\underline{\mathbf{X}}$ such that, for each $n \in \mathbb{N}$, each of the basis sets U_1, U_2, \dots, U_n is a union of θ_n -classes. Now $\bigcap \{\theta_n \mid n \in \mathbb{N}\} = \Delta_X$. By Lemma 1.5, we can define the

clopen congruence $\sigma_n := \text{Syn}(\theta_n)$ on $\underline{\mathbf{X}}$, for each $n \in \mathbb{N}$. It is easy to check that $\sigma_1 \supseteq \sigma_2 \supseteq \sigma_3 \supseteq \dots$ and that $\bigcap \{\sigma_n \mid n \in \mathbb{N}\} = \Delta_X$.

Define $\mathcal{S} := \{\sigma_1, \sigma_2, \sigma_3, \dots\}$. In order to apply Lemma 2.2, let $r \in R \cup \{=\}$ and let $(a_1, \dots, a_m) \in X^m \setminus r^{\mathbf{X}}$. Since $r^{\mathbf{X}}$ is topologically closed, there are clopen sets V_1, \dots, V_m such that $a_i \in V_i$ and $r^{\mathbf{X}} \cap (V_1 \times \dots \times V_m) = \emptyset$. For $i = 1, \dots, m$, we have $\bigcap \{a_i/\sigma_n \mid n \in \mathbb{N}\} = \{a_i\} \subseteq V_i$. By compactness, there is a $k \in \mathbb{N}$ such that $a_i/\sigma_k \subseteq V_i$, for all $i = 1, \dots, m$. This gives $(a_1/\sigma_k, \dots, a_m/\sigma_k) \notin r^{\mathbf{X}/\sigma_k}$. By Lemma 2.2, it follows that $\underline{\mathbf{X}}$ is isomorphic to a surjective inverse limit of finite structures, indexed over \mathbb{N} . Using the Separation Theorem 1.3, we can check that $\underline{\mathbf{X}}$ is pointwise non-separable with respect to \mathcal{M} , as $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$. \square

We next extend Proposition 3.12 to a partial converse for the TILT 3.10. We need to assume that the witness to non-standardness $\underline{\mathbf{X}}$ has a very special topology, which we now describe.

Let $\lambda \in \mathbb{N}$. A λ -**compactification** of a set X is an idempotent map $f : X \rightarrow X$ with $|f(X)| \leq \lambda$. Given a set X and a λ -compactification $f : X \rightarrow X$, define \mathcal{J}^f to be the topology on X consisting of the subsets U of X such that, for each $a \in U \cap f(X)$, the set $U \cap f^{-1}(a)$ is cofinite in $f^{-1}(a)$. Then a topological space $\underline{\mathbf{X}}$ is a λ -**space** if there is a λ -compactification $f : X \rightarrow X$ such that $\underline{\mathbf{X}} = \langle X; \mathcal{J}^f \rangle$.

Lemma 3.13. *Let $\underline{\mathbf{X}}$ be a topological space and let $\lambda \in \mathbb{N}$. Then the following are equivalent:*

- (i) $\underline{\mathbf{X}}$ is a λ -space;
- (ii) $\underline{\mathbf{X}}$ is a disjoint union of at most λ subspaces, each of which is either a finite discrete space or a one-point compactification of an infinite discrete space;
- (iii) $\underline{\mathbf{X}}$ is a Boolean space with at most λ non-isolated points.

Theorem 3.14. *Assume that R is finite. Let $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$ witness that a uH -class \mathcal{M} is non-standard. Assume that \mathbf{X} is countable, locally finite and has FDSC, and that $\underline{\mathbf{X}}$ has a λ -space topology, for some $\lambda \in \mathbb{N}$. Then $\underline{\mathbf{X}}$ is isomorphic to a surjective inverse limit $\underline{\mathbf{Z}}$ of finite structures, indexed over \mathbb{N} , that satisfies the conditions of the TILT 3.10; namely, $\underline{\mathbf{Z}}$ is pointwise non-separable with respect to \mathcal{M} and $\text{T}(\underline{\mathbf{Z}}, \mathcal{M})$ holds.*

Proof. By Proposition 3.12, we may assume that $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$, a surjective inverse limit of finite structures that is pointwise non-separable with respect to \mathcal{M} .

Let Λ be the (finite, non-empty) set of non-isolated points of $\underline{\mathbf{X}}$. For each $n \in \mathbb{N}$, define $\Lambda_n := \pi_n(\Lambda) \subseteq X_n$ and $V_n := \pi_n^{-1}(\Lambda_n) \subseteq X$. Then $V_1 \supseteq V_2 \supseteq V_3 \supseteq \dots \supseteq \Lambda$ in $\underline{\mathbf{X}}$. Since each projection π_n is continuous, each V_n is clopen in $\underline{\mathbf{X}}$. As V_n contains Λ , it follows that V_n is cofinite in $\underline{\mathbf{X}}$. Now define $V'_n := X \setminus V_n$, for all $n \in \mathbb{N}$. We have a sequence $V'_1 \subseteq V'_2 \subseteq V'_3 \subseteq \dots$ of finite subsets of X , each one disjoint from Λ .

We want to establish that $\text{T}(\underline{\mathbf{X}}, \mathcal{M})$ holds. So define

$$Y_{n+1} := \pi_{n+1}(\Lambda \cup V'_n) \subseteq X_{n+1},$$

for each $n \in \mathbb{N}$. (Note that we don't need to worry about Y_1 , since we can remove \mathbf{X}_1 from the inverse system without changing the limit.) For all $n \in \mathbb{N}$, we have

$$Y_{n+1} = \pi_{n+1}(\Lambda \cup V'_n) = \pi_{n+1}(\Lambda) \cup \pi_{n+1}(X \setminus V_n) = \Lambda_{n+1} \cup \varphi_n^{-1}(X_n \setminus \Lambda_n),$$

and therefore $\varphi_n(X_{n+1} \setminus Y_{n+1}) \subseteq \Lambda_n$. We also have $\varphi_n(\Lambda_{n+1}) \subseteq \Lambda_n$, for each $n \in \mathbb{N}$. It follows by induction that $\varphi_{mn}(X_m \setminus Y_m) \subseteq Y_n$, for all $m > n > 1$.

We would now like to show that $\text{Sg}_{\mathbf{X}_{n+1}}(Y_{n+1}) \in \mathcal{M}$, for all $n \in \mathbb{N}$. We have

$$\text{Sg}_{\mathbf{X}_{n+1}}(Y_{n+1}) = \text{Sg}_{\mathbf{X}_{n+1}}(\pi_{n+1}(\Lambda \cup V'_n)) = \pi_{n+1}(\text{Sg}_{\mathbf{X}}(\Lambda \cup V'_n)),$$

for each $n \in \mathbb{N}$, and so we define

$$\mathbf{W}_n := \text{Sg}_{\mathbf{X}}(\Lambda \cup V'_n).$$

Since \mathbf{X} is locally finite and $V'_1 \subseteq V'_2 \subseteq V'_3 \subseteq \dots$ is a sequence of finite subsets of X , we have constructed a sequence $\mathbf{W}_1 \leq \mathbf{W}_2 \leq \mathbf{W}_3 \leq \dots$ of finite substructures of \mathbf{X} . As $\underline{\mathbf{X}} \in \mathcal{M}_{\text{Bt}}$, each \mathbf{W}_n belongs to \mathcal{M} .

We have $\text{Sg}_{\mathbf{X}_{n+1}}(Y_{n+1}) = \pi_{n+1}(\mathbf{W}_n)$ and $\mathbf{W}_n \in \mathcal{M}$, for every $n \in \mathbb{N}$. We would like to prove that each $\text{Sg}_{\mathbf{X}_{n+1}}(Y_{n+1})$ belongs to \mathcal{M} by showing that each restriction $\pi_{n+1}|_{\mathbf{W}_n} : \mathbf{W}_n \rightarrow \mathbf{X}_{n+1}$ is an embedding. Unfortunately, this need not always be true. But we will be able to show that, without loss of generality, we can assume that each $\pi_{n+1}|_{\mathbf{W}_n}$ is an embedding.

First notice that, since $\underline{\mathbf{X}}$ is the inverse limit and R is finite, for every finite substructure \mathbf{W} of \mathbf{X} , there is an $n \in \mathbb{N}$ such that $\pi_n|_{\mathbf{W}} : \mathbf{W} \rightarrow \mathbf{X}_n$ is an embedding.

We now define a map $c : \mathbb{N} \rightarrow \mathbb{N}$ as follows. Let $c(1) := 1$. Given $c(n)$, for some $n \in \mathbb{N}$, we can choose $c(n+1) > c(n)$ so that $\pi_{c(n+1)}$ embeds $\mathbf{W}_{c(n)}$ into $\mathbf{X}_{c(n+1)}$. The definition of inverse limit assures us that $\underline{\mathbf{X}} \cong \varprojlim \{\mathbf{X}_{c(n)} \mid n \in \mathbb{N}\}$. So we can assume, without loss of generality, that in fact $\underline{\mathbf{X}} = \varprojlim \{\mathbf{X}_{c(n)} \mid n \in \mathbb{N}\}$, whence $c = \text{id}_{\mathbb{N}}$. Now, for each $n \in \mathbb{N}$, we have

$$\text{Sg}_{\mathbf{X}_{n+1}}(Y_{n+1}) = \pi_{n+1}(\mathbf{W}_n) \cong \mathbf{W}_n \in \mathcal{M}.$$

Hence $\text{T}(\underline{\mathbf{X}}, \mathcal{M})$ holds (at least for all $n > 1$). \square

We do not know the true extent of the applicability of the TILT 3.10. But most of the applications of the FILT 3.5 and the SILT 3.9 in this paper produce an inverse limit that satisfies all of the conditions of Theorem 3.14, and therefore could also be realized as an application of the TILT 3.10. We will see applications of the FILT 3.5 and the SILT 3.9 to lattices, ordered sets, graphs, unary algebras and semigroups. Our preferred technique will always be the SILT 3.9, since it has the added bonus of excluding finite axiomatizability.

4. LATTICES

The inverse-limit techniques of Section 3 give us methods for showing not only that a uH-class \mathcal{M} is non-standard, but also that its Boolean core \mathcal{M}_{Bc} is not even universally definable. This leads us to ask the baseline question about the topological prevariety $\mathcal{M}_{\text{Bc}} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathcal{M}_{\text{fin}})$: Is it first-order definable at all? In this section, we will prove that, for every finite structure of finite type \mathbf{M} with a lattice reduct, the topological prevariety $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ is first-order definable. Then we will apply the SILT 3.9 to the uH-class generated by a particular finite lattice, to obtain a non-standard uH-class whose Boolean core is nevertheless first-order definable.

The proof of our first lemma actually shows that every compact Hausdorff topological meet-semilattice has a least element.

Lemma 4.1. *Let \mathbf{M} be a finite structure with a lattice reduct, let $\underline{\mathbf{X}} \in [\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})]_{\text{Bt}}$ and let $\varphi : \underline{\mathbf{X}} \rightarrow \mathbf{M}$ be a continuous homomorphism. Then, for all $a \in \varphi(X)$, there are $d \leq u$ in $\underline{\mathbf{X}}$ such that $\varphi^{-1}(a) = [d, u]$.*

Proof. There are operations $\vee, \wedge \in G$ such that $\langle M; \vee^M, \wedge^M \rangle$ is a lattice and $\langle X; \vee^X, \wedge^X, \mathcal{T} \rangle$ is a topological lattice. For each $y \in X$, the set $\downarrow y$ is closed in \mathfrak{X} . (To see this, define $e : \mathfrak{X} \rightarrow \mathfrak{X}$ by $e(x) = x \wedge y$. Then e is continuous, as \wedge is continuous, and so e is closed, as \mathfrak{X} is compact and Hausdorff. Now $\downarrow y = e(X)$, which is closed.)

Now let $a \in \varphi(X)$. Then $\varphi^{-1}(a)$ is closed under \vee and \wedge . This means that we need only show that $\varphi^{-1}(a)$ has a (necessarily unique) minimal element d and maximal element u . Suppose it has no minimal element. For all $y \in \varphi^{-1}(a)$, the set $U_y := \varphi^{-1}(a) \setminus \downarrow y$ is open. For all $x \in \varphi^{-1}(a)$, there is a $y < x$ in $\varphi^{-1}(a)$, and so $x \in U_y$. Thus $\{U_y \mid y \in \varphi^{-1}(a)\}$ covers $\varphi^{-1}(a)$. Let U_{y_1}, \dots, U_{y_n} be a finite subcover, and define $z := y_1 \wedge y_2 \wedge \dots \wedge y_n \in \varphi^{-1}(a)$. There must be some $i \in \{1, \dots, n\}$ with $z \in U_{y_i}$, and therefore $z \not\leq y_i$. But this contradicts the definition of z . Thus $\varphi^{-1}(a)$ does have a minimal element d and, similarly, a maximal element u . \square

Theorem 4.2. *Let \mathbf{M} be a finite structure of finite type that has a lattice reduct. Then $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$ is first-order definable.*

Proof. The type $G \cup R$ of \mathbf{M} is finite. So there is a first-order sentence α , in the language of \mathbf{M} , asserting that, for each $r \in R \cup \{=\}$ and $(x_1, \dots, x_m) \notin r$, there exists a set of at most $|M|$ ordered pairs (d, u) , with $d \leq u$, such that the intervals $[d, u]$ form the kernel of a homomorphism φ into \mathbf{M} with $(\varphi(x_1), \dots, \varphi(x_m)) \notin r^{\mathbf{M}}$. There is a first-order sentence β asserting that a structure is isomorphic to the total 1-element structure $\mathbf{1} = \mathbf{M}^\emptyset$. We define $\Sigma := \{\alpha\}$, if $\mathbf{1}$ embeds into \mathbf{M} , and we define $\Sigma := \{\alpha, \neg\beta\}$, otherwise. We claim that $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M}) = \text{Mod}_{\text{Bt}}(\Sigma)$.

Let $\mathfrak{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M}) \subseteq [\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})]_{\text{Bt}}$. Then \mathfrak{X} has an underlying Boolean topological lattice and, by Lemma 4.1 and the Separation Theorem 1.3, the separating morphisms into \mathbf{M} provide what is needed to satisfy α . Thus $\mathfrak{X} \in \text{Mod}_{\text{Bt}}(\Sigma)$.

Now let $\mathfrak{X} \in \text{Mod}_{\text{Bt}}(\Sigma)$. To see that $\mathfrak{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$, assume that $\mathfrak{X} \not\cong \mathbf{1}$ and choose $r \in R \cup \{=\}$ and $(x_1, \dots, x_m) \in X^m \setminus r^{\mathbf{X}}$. Since \mathfrak{X} satisfies α , there exists a homomorphism $\varphi : \mathbf{X} \rightarrow \mathbf{M}$ such that $(\varphi(x_1), \dots, \varphi(x_m)) \notin r^{\mathbf{M}}$. It only remains to show that φ is continuous. Let $\varphi(X) = \{a_1, \dots, a_k\} \subseteq M$. From α , we know that there are $d_1, \dots, d_k, u_1, \dots, u_k \in X$ such that $d_i \leq u_i$ and

$$\varphi^{-1}(a_i) = [d_i, u_i] = \uparrow d_i \cap \downarrow u_i,$$

for $i = 1, \dots, k$. We already know that \mathbf{X} is separated by homomorphisms into \mathbf{M} . So $\mathfrak{X} \in [\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})]_{\text{Bt}}$, and therefore \mathfrak{X} has a Boolean topological lattice reduct. This implies that each $\uparrow d_i$ and $\downarrow u_i$ is closed. Thus each $\varphi^{-1}(a_i)$ is closed. We have partitioned \mathfrak{X} into finitely many closed sets: $\{\varphi^{-1}(a_i) \mid i \in \{1, \dots, k\}\}$. So each $\varphi^{-1}(a_i)$ must be clopen, whence φ is continuous. \square

We now give an example of an application of the SILT 3.9 to the uH-class generated by a particular finite lattice. Belkin [2] proved that the 10-element lattice \mathbf{M}_{3-3} , given in Figure 2, has no finite uH-axiomatization. (See also Gorbunov [23] for a full proof.) Conveniently, Belkin's proof provides exactly the two conditions we need to apply the SILT 3.9. For each $n \in \mathbb{N}$, Belkin defines a finite lattice \mathbf{X}_n , shown on the right in Figure 2, and establishes two facts:

- (B1) every homomorphism from \mathbf{X}_n into \mathbf{M}_{3-3} identifies a and b ;
- (B2) every n -generated sublattice of \mathbf{X}_n lies in $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M}_{3-3})$.

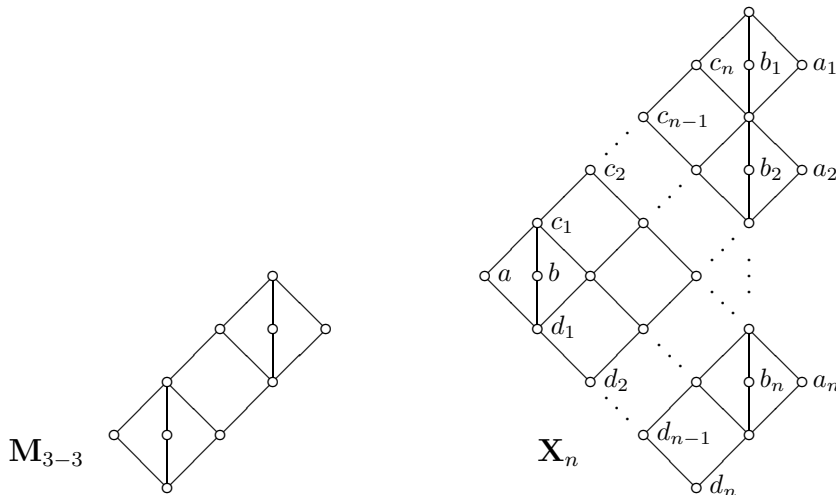


FIGURE 2

Example 4.3. Let \mathbf{M}_{3-3} be the lattice in Figure 2. The uH -class $\mathbb{ISP}^+(\mathbf{M}_{3-3})$ is neither standard nor finitely axiomatizable, and its Boolean core $\mathbb{IS}_{cP}^+(\mathbf{M}_{3-3})$ is first-order definable but not universally definable.

Proof. For each $n \in \mathbb{N}$, we can define a surjective connecting homomorphism $\varphi_n : \mathbf{X}_{n+1} \rightarrow \mathbf{X}_n$ as follows. First check that the congruence $\text{Cg}_{\mathbf{X}_{n+1}}(a_{n+1}, b_{n+1})$ on \mathbf{X}_{n+1} , generated by identifying a_{n+1} with b_{n+1} , produces a lattice isomorphic to \mathbf{X}_n . We choose φ_n to be the composition of this isomorphism with the natural map of $\text{Cg}_{\mathbf{X}_{n+1}}(a_{n+1}, b_{n+1})$. Let $\mathbf{X} := \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ be the resulting inverse limit, which must be surjective by Lemma 2.1.

Define $\mathcal{M} := \mathbb{ISP}^+(\mathbf{M}_{3-3})$. Then $\mathcal{M}_{\text{Bc}} = \mathbb{IS}_{cP}^+(\mathbf{M}_{3-3})$. Item (B1) gives us the pointwise non-separability of \mathbf{X} with respect to \mathcal{M} . Item (B2) gives us $S(\mathbf{X}, \mathcal{M})$. Applying the SILT 3.9, it follows that \mathcal{M} is neither standard nor finitely axiomatizable. Theorem 2.17 says that \mathcal{M}_{Bc} is not universally definable. Theorem 4.2 says that \mathcal{M}_{Bc} is nevertheless first-order definable. \square

5. ULTRAPRODUCT TECHNIQUES

In this section, we present two general techniques for showing that a topological prevariety is not first-order definable at all. In later sections, the first technique will be applied to various ordered structures, and the second technique to graphs, unary algebras and cyclic semigroups. The following extension of Example 2.16 illustrates the role that ultraproducts will play.

Example 5.1. Let \mathcal{M} denote the uH -class of all unars. Then the Boolean core $\mathcal{M}_{\text{Bc}} = \mathbb{IS}_{cP}^+(\mathcal{M}_{\text{fin}})$ is not even first-order definable.

Proof. As in Example 2.16, define the finite unar $\mathbf{U}_n := (\{0, 1, \dots, n-1\} \cup \{\infty\}; s)$, where $s(i) := i \oplus_n 1$ and $s(\infty) := \infty$. Let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} , and consider the ultraproduct $\mathbf{Z} := \prod_{\mathcal{U}} \{\mathbf{U}_n \mid n \in \mathbb{N}\}$. Since ultraproduct preserves first-order properties, the operation s of \mathbf{Z} is bijective and has a unique fixed point ∞ . For each $k \in \mathbb{N}$, the sentence $(\forall x)(s^k(x) \approx x \implies s(x) \approx x)$ holds on a

cofinite subset of $\{\mathbf{U}_n \mid n \in \mathbb{N}\}$, and so it holds in \mathbf{Z} . Thus \mathbf{Z} is a disjoint union of the fixed point ∞ and copies of the integers \mathbb{Z} with successor. So the operation s of \mathbf{Z} is continuous with respect to the topology \mathcal{T}^f given by the 1-compactification $f : Z \rightarrow \{\infty\}$. Since $\{\mathbf{U}_n \mid n \in \mathbb{N}\} \subseteq \mathcal{M}_{\text{Bc}}$, we conclude that $\mathcal{Z} := \langle Z; s, \mathcal{T}^f \rangle$ is a Boolean topological model of the first-order theory of \mathcal{M}_{Bc} . However, it is easy to see that the only clopen congruence on \mathcal{Z} is the universal congruence. Thus $\mathcal{Z} \notin \mathcal{M}_{\text{Bc}}$, by the Separation Theorem 1.3. \square

For a class \mathcal{K} of structures, we denote by $\text{Th}(\mathcal{K})$ the set of all first-order sentences that hold in every member of \mathcal{K} . Given a uH-class \mathcal{M} , we would like to demonstrate that \mathcal{M}_{Bc} is not first-order definable by exhibiting a Boolean topological structure \mathcal{Z} that is a model of $\text{Th}(\mathcal{M}_{\text{Bc}})$ but is not in \mathcal{M}_{Bc} ; in symbols,

$$(\diamond) \quad \mathcal{Z} \in \text{Mod}_{\text{Bt}}(\text{Th}(\mathcal{M}_{\text{Bc}})) \quad \text{and} \quad \mathcal{Z} \notin \mathcal{M}_{\text{Bc}}.$$

Usually, we cannot construct the witness \mathcal{Z} to non-definability as directly as we did in Example 5.1. Instead, we will construct the witness in two stages.

For both of our techniques, we begin with a structure $\mathbf{X} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$, already known to witness the non-standardness of \mathcal{M} . We use \mathbf{X} to construct an ultraproduct \mathbf{Z} of members of $\mathcal{M}_{\text{fin}} \subseteq \mathcal{M}_{\text{Bc}}$, thereby ensuring that \mathbf{Z} is a model of $\text{Th}(\mathcal{M}_{\text{Bc}})$ and giving us

$$(\heartsuit) \quad \mathbf{Z} \in \text{Mod}(\text{Th}(\mathcal{M}_{\text{Bc}})) \quad \text{and} \quad \mathbf{X} \notin \mathcal{M}_{\text{Bc}}.$$

Comparing \diamond and \heartsuit suggests two natural avenues for producing the desired witness to non-definability. Our first technique will replace \mathbf{Z} on the left of \heartsuit with \mathbf{X} , by establishing that \mathbf{Z} is elementarily equivalent to \mathbf{X} . Our second technique will replace \mathbf{X} on the right with \mathcal{Z} , by producing a compatible Boolean topology on \mathbf{Z} and then showing that \mathcal{Z} is also not in \mathcal{M}_{Bc} .

Our first technique applies to uH-classes whose operations are all at most unary. In order to describe this technique, first consider a structure \mathbf{X} whose operations are at most unary, and let \mathbf{X}_0 denote the substructure of \mathbf{X} consisting of all the values of constant term functions of \mathbf{X} . Now let \mathbf{Y} be a substructure of \mathbf{X} with $\mathbf{X}_0 < \mathbf{Y} \leq \mathbf{X}$. We say that \mathbf{Y} is **isolated** in \mathbf{X} if

- the set $Y' := (X \setminus Y) \cup X_0$ forms a substructure \mathbf{Y}' of \mathbf{X} , and
- for each $r \in R$, we have $r^{\mathbf{X}} = r^{\mathbf{Y}} \cup r^{\mathbf{Y}'}$.

To elucidate this concept, define the graph $\mathbf{G}_X = \langle X \setminus X_0; \sim \rangle$ such that $a \sim b$ if at least one of the following conditions holds:

- for some unary $g \in G$, we have $g^{\mathbf{X}}(a) = b$ or $g^{\mathbf{X}}(b) = a$;
- for some $r \in R$ and $(c_1, \dots, c_m) \in r^{\mathbf{X}}$, we have $\{a, b\} \subseteq \{c_1, \dots, c_m\}$.

Then it is easy to see that \mathbf{Y} is a minimal isolated substructure of \mathbf{X} if and only if $Y \setminus X_0$ is a connected component of the graph \mathbf{G}_X .

First Ultraproduct Technique 5.2. *Let \mathcal{M} be a non-standard uH-class, with witness $\mathbf{X} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$. Assume that*

- (i) *the type of \mathcal{M} is finite, with all operations at most unary,*
- (ii) *there are only finitely many values of constant term functions of \mathbf{X} , and*
- (iii) *up to isomorphism, there are only finitely many minimal isolated substructures of \mathbf{X} , all of which are finite.*

Then \mathcal{M}_{Bc} is not first-order definable.

Proof. Let \mathbf{X}_0 be the substructure of \mathbf{X} formed by the set of all values of constant term functions of \mathbf{X} . Let $\mathbf{V}_1, \dots, \mathbf{V}_k, \mathbf{W}_1, \dots, \mathbf{W}_\ell$ be fixed representatives of the distinct isomorphism types of the minimal isolated substructures of \mathbf{X} , so that \mathbf{X} consists of exactly $m_i \in \mathbb{N}$ copies of \mathbf{V}_i together with infinitely many copies of \mathbf{W}_j , for $1 \leq i \leq k$ and $1 \leq j \leq \ell$. For each $n \in \mathbb{N}$, define the substructure \mathbf{X}_n of \mathbf{X} to be a particular union of all m_i copies of each \mathbf{V}_i and n copies of each \mathbf{W}_j .

Now let \mathcal{U} be a non-principal ultrafilter on \mathbb{N} , and define the ultraproduct $\mathbf{Z} := \prod_{\mathcal{U}} \{\mathbf{X}_n \mid n \in \mathbb{N}\}$. As $\{\mathbf{X}_n \mid n \in \mathbb{N}\} \subseteq \mathcal{M}_{\text{fin}} \subseteq \mathcal{M}_{\text{Bc}}$, we have $\mathbf{Z} \in \text{Mod}(\text{Th}(\mathcal{M}_{\text{Bc}}))$. The proof will be finished if we can establish that \mathbf{X} is elementarily equivalent to \mathbf{Z} , for then we will have $\mathfrak{X} \in \text{Mod}_{\text{Bt}}(\text{Th}(\mathcal{M}_{\text{Bc}})) \setminus \mathcal{M}_{\text{Bc}}$.

First we note that, as the type of \mathcal{M} is finite, there is a set Σ of first-order sentences asserting that:

- the substructure of all values of constant term functions is isomorphic to \mathbf{X}_0 ;
- each \mathbf{X}_n is embeddable as an isolated substructure;
- the union of $m_i + 1$ isolated copies of \mathbf{V}_i is not embeddable as an isolated substructure;
- every element is in an isolated copy of some \mathbf{V}_i or \mathbf{W}_j .

Clearly $\mathbf{X} \models \Sigma$. Each sentence of Σ holds in all but finitely many members of $\{\mathbf{X}_n \mid n \in \mathbb{N}\}$, and therefore holds in the ultraproduct \mathbf{Z} . Thus $\mathbf{Z} \models \Sigma$.

We claim that Σ is \aleph_0 -categorical. Let \mathbf{Y}_1 and \mathbf{Y}_2 be countably infinite models of Σ . Then, for both \mathbf{Y}_1 and \mathbf{Y}_2 , the substructure formed by the values of constant term functions is isomorphic to \mathbf{X}_0 . Consider the decompositions of \mathbf{Y}_1 and \mathbf{Y}_2 into minimal isolated substructures. From Σ , these decompositions both include exactly m_i copies of each \mathbf{V}_i together with \aleph_0 copies of each \mathbf{W}_j , and nothing else. It follows that $\mathbf{Y}_1 \cong \mathbf{Y}_2$, and consequently Σ is \aleph_0 -categorical.

The infinite structures \mathbf{X} and \mathbf{Z} are models of the theory Σ , which is of finite type. As Σ is \aleph_0 -categorical, it follows by the Downward Löwenheim-Skolem Theorem that \mathbf{X} and \mathbf{Z} must be elementarily equivalent, as required. \square

For our second ultraproduct technique, we need to find a compatible Boolean topology on an ultraproduct \mathbf{Z} of members of \mathcal{M}_{Bc} such that the topological structure $\underline{\mathbf{Z}}$ does not belong to \mathcal{M}_{Bc} . In the absence of special information about the ultraproduct \mathbf{Z} , our Example 5.1 suggests that we look for a compatible λ -space topology on \mathbf{Z} . It turns out that this can often be done in a quite general setting.

Consider sets X and Y and maps $f : X \rightarrow X$ and $g : Y \rightarrow Y$. Notice that, if f is a λ -compactification of X and $\langle Y; g \rangle$ is a model of the universal theory of $\langle X; f \rangle$, then g is a λ -compactification of Y . For a structure \mathbf{X} and map $f : X \rightarrow X$, we will use $\langle \mathbf{X}; f \rangle$ to denote the structure obtained from \mathbf{X} by adding the new unary operation f .

Second Ultraproduct Technique 5.3. *Let \mathcal{M} be a non-standard uH-class, with witness $\mathfrak{X} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$. Assume that*

- (i) *the structure \mathbf{X} is locally finite,*
- (ii) *there is a λ -compactification $f : X \rightarrow X$, for some $\lambda \in \mathbb{N}$, such that \mathfrak{X} has the corresponding λ -space topology, and*
- (iii) *each model $\langle \mathbf{Y}; g \rangle$ of the universal theory of $\langle \mathbf{X}; f \rangle$ is a Boolean topological structure under the λ -space topology \mathcal{T}^g .*

Then \mathcal{M}_{Bc} is not first-order definable.

Proof. By Burris and Sankappanavar [6, V.2.14], the structure $\langle \mathbf{X}; f \rangle$ can be embedded in an ultraproduct $\langle \mathbf{Z}; g \rangle$ of its finitely generated substructures, all of which are finite. Since each substructure of $\langle \mathbf{X}; f \rangle$ is a model of the universal theory of $\langle \mathbf{X}; f \rangle$, it follows that $\langle \mathbf{Z}; g \rangle$ is a model as well. So $g : Z \rightarrow Z$ is a λ -compactification of Z . By assumption (iii), we can define the Boolean topological structure \mathcal{Z} to be the structure \mathbf{Z} enriched with the λ -space topology \mathcal{T}^g . We now have $\mathcal{Z} \in \text{Mod}_{\text{Bt}}(\text{Th}(\mathcal{M}_{\text{Bc}}))$, since each finite substructure of \mathbf{X} is in \mathcal{M}_{Bc} .

We want to prove that $\mathcal{Z} \notin \mathcal{M}_{\text{Bc}}$. There is an embedding $\eta : \langle \mathbf{X}; f \rangle \rightarrow \langle \mathbf{Z}; g \rangle$. So \mathbf{X} is isomorphic to a topologically closed substructure of \mathcal{Z} . As $\mathbf{X} \notin \mathcal{M}_{\text{Bc}}$ and the class \mathcal{M}_{Bc} is closed under \mathbb{I} and \mathbb{S}_c , we must have $\mathcal{Z} \notin \mathcal{M}_{\text{Bc}}$. \square

6. ORDERED STRUCTURES

In this section, we will apply our First Ultraproduct Technique 5.2 to a number of interesting and historically significant topological quasivarieties of ordered structures. Our examples stem from Priestley's 1970 duality for the quasivariety of bounded distributive lattices [33], which sparked the discovery of a series of natural dualities for closely related quasivarieties: De Morgan algebras (Cornish and Fowler [14], 1977), median algebras (Isbell [25], 1980, and Werner [38], 1981), Stone and double Stone algebras (Davey [15], 1978, and Davey [16], 1982) and Kleene algebras (Davey and Werner [20], 1983). In all of these dualities, the dual category is the Boolean core of a uH-class of ordered sets with some additional structure. (These results can all be found in Section 4.3 of [8].) Each of these dual categories is known to be non-standard [10]. We will use the FUPT 5.2 to show that, in fact, none of these dual categories is even first-order definable.

Stralka [35] discovered the original example of a non-standard uH-class, which we now review. Define $\mathbf{2} := \langle \{0, 1\}; \leq \rangle$ to be the 2-element chain. By the non-topological version of the Separation Theorem 1.3, the uH-class of all ordered sets is $\mathcal{O} := \mathbb{ISP}^+(\mathbf{2})$. The Boolean core, $\mathcal{O}_{\text{Bc}} = \mathbb{IS}_c\mathbb{P}^+(\mathbf{2})$, is the category of Priestley spaces, the dual to bounded distributive lattices under Priestley duality [33].

Now let $\langle C; \mathcal{T} \rangle$ be the Cantor space, obtained by removing open middle thirds from the unit interval: from $[0, 1]$ we remove the open interval $(\frac{1}{3}, \frac{2}{3})$, then $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, and so forth. The remaining set C has a countable sequence of covering pairs under the usual order: $\frac{1}{3} < \frac{2}{3}$, $\frac{1}{9} < \frac{2}{9}$, $\frac{7}{9} < \frac{8}{9}$, \dots . We define the **Stralka order** \leq_S on C by $x \leq_S y$ if either $x = y$ or x is covered by y in the usual order. Since \leq_S is closed in $C \times C$, the **Stralka space** $\mathcal{S} := \langle C; \leq_S, \mathcal{T} \rangle$ is in \mathcal{O}_{Bt} , and Stralka [35] showed that $\mathcal{S} \notin \mathcal{O}_{\text{Bc}}$. We now show that the Stralka space \mathcal{S} can also be realized as an inverse limit in an application of the FILT 3.5.

Example 6.1. *The uH-class \mathcal{O} of all ordered sets is non-standard.*

Proof. For each $n \in \mathbb{N}$, define the structure $\mathbf{S}_n := \langle \{0, 1, \dots, 2^n - 1\}; \preceq \rangle$, where \preceq is the 'covering or equal to' relation given by $i \preceq j$ if and only if $j \in \{i, i+1\}$. Then $\{\mathbf{S}_n \mid n \in \mathbb{N}\}$ forms a surjective inverse system, with each connecting morphism $\varphi_n : \mathbf{S}_{n+1} \rightarrow \mathbf{S}_n$ given by $\varphi_n(i) := \lfloor i/2 \rfloor$. It is easy to check that the inverse limit \mathbf{X} of this system is isomorphic to the Stralka space \mathcal{S} .

We now check the conditions of the FILT 3.5. For each $n > 1$, we have $0 \not\preceq 2^n - 1$ in \mathbf{S}_n . However, each homomorphism $\psi : \mathbf{S}_n \rightarrow \mathbf{M} \in \mathcal{O}$ satisfies $\psi(0) \leq \psi(2^n - 1)$. Thus \mathbf{X} is pointwise non-separable with respect to \mathcal{O} . To verify $F(\mathbf{X}, \mathcal{O})$, notice

that, if φ_n is one-to-one on a substructure \mathbf{Y} of \mathbf{S}_{n+1} , then \preceq is transitive on \mathbf{Y} and consequently $\mathbf{Y} \in \mathcal{O}$. By the FILT 3.5, we have $\underline{\mathfrak{S}} \cong \underline{\mathfrak{X}} \in \mathcal{O}_{\text{Bt}} \setminus \mathcal{O}_{\text{Bc}}$. \square

Example 6.2. *The category \mathcal{O}_{Bc} of Priestley spaces is not first-order definable.*

Proof. By the previous example, we have $\underline{\mathfrak{S}} \in \mathcal{O}_{\text{Bt}} \setminus \mathcal{O}_{\text{Bc}}$. There are no operations in the type of $\underline{\mathfrak{S}}$ and, up to isomorphism, the structure \mathbf{S} has exactly two minimal isolated substructures, a 1-element chain and a 2-element chain. So we conclude from the FUPT 5.2 that \mathcal{O}_{Bc} is not first-order definable. \square

We note that Example 6.2 cannot be derived from the SUPT 5.3, as Bezhanishvili, Mines and Morandi [5] have shown that any closed order on a compact scattered space (and therefore on a λ -space) does form a Priestley space.

Similar arguments apply to the uH-class \mathcal{E} of all sets equipped with an equivalence relation and the uH-class \mathcal{Q} of all quasi-ordered sets (reflexive and transitive).

Example 6.3.

- (i) *The uH-class \mathcal{E} of all sets with an equivalence relation is non-standard, and the category \mathcal{E}_{Bc} is not even first-order definable.*
- (ii) *The uH-class \mathcal{Q} of all quasi-ordered sets is non-standard, and the category \mathcal{Q}_{Bc} is not even first-order definable.*

Proof. Based on the Stralka space $\langle C; \leq_S, \mathcal{T} \rangle$, define the equivalence \equiv_S on C by $\equiv_S := \leq_S \cup \geq_S$. Then \equiv_S is closed in $C \times C$. So the structure $\underline{\mathfrak{X}} := \langle C; \equiv_S, \mathcal{T} \rangle$ is in \mathcal{E}_{Bt} and therefore also in \mathcal{Q}_{Bt} . We have $0 \not\equiv_S 1$ in $\underline{\mathfrak{X}}$. But, for each $\mathbf{M} = \langle M; \trianglelefteq \rangle$ in \mathcal{Q}_{fin} , every continuous homomorphism $\psi : \underline{\mathfrak{X}} \rightarrow \mathbf{M}$ satisfies $\psi(0) \trianglelefteq \psi(1)$. Thus $\underline{\mathfrak{X}} \notin \mathcal{Q}_{\text{Bc}}$, and therefore $\underline{\mathfrak{X}} \notin \mathcal{E}_{\text{Bc}}$. So both \mathcal{E} and \mathcal{Q} are non-standard. Up to isomorphism, the structure \mathbf{X} has exactly two minimal isolated substructures, a 1-element one and a 2-element one. So we can conclude from the FUPT 5.2 that neither \mathcal{E}_{Bc} nor \mathcal{Q}_{Bc} is first-order definable. \square

Consider any finite quasi-ordered set $\mathbf{Q} = \langle Q; \trianglelefteq \rangle$ such that \trianglelefteq is neither the universal relation nor the equality relation. It is straightforward to check that the uH-class $\mathbb{ISP}^+(\mathbf{Q})$ is either \mathcal{O} , \mathcal{E} or \mathcal{Q} .

Corollary 6.4. *Let $\mathbf{Q} = \langle Q; \trianglelefteq \rangle$ be a finite quasi-ordered set. Then $\mathbb{ISP}^+(\mathbf{Q})$ is standard if and only if \trianglelefteq is either the universal relation or the equality relation.*

We now look at ordered sets with additional structure, building on some results from [10].

Example 6.5. *The category $\mathbb{IS}_c\mathbb{P}^+(\mathbf{DM})$, strongly dual to De Morgan algebras, is not first-order definable.*

Proof. By Cornish and Fowler [14] ([8, 4.3.16]), we have $\mathbf{DM} := \langle \{0, a, b, 1\}; f, \leq \rangle$, where \leq is the non-linear lattice order with a at the top and b at the bottom, and f is the bijection fixing 0 and 1 and interchanging a and b . Define $\mathbf{M} := \mathbb{ISP}^+(\mathbf{DM})$ and $\underline{\mathfrak{X}} := \langle C; f, \leq_S, \mathcal{T} \rangle$, where $\langle C; \leq_S, \mathcal{T} \rangle$ is the Stralka space and $f(x) := 1 - x$. In Example 2.11 of [10], it is shown that $\underline{\mathfrak{X}} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$. The result follows from the FUPT 5.2, as every minimal isolated substructure of \mathbf{X} is isomorphic to the substructure with underlying set $\{0, 1\}$, $\{\frac{1}{3}, \frac{2}{3}\}$ or $\{\frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}\}$. \square

Example 6.6. *The category $\mathbb{IS}_c\mathbb{P}^+(\mathbf{M})$, strongly dual to median algebras, is not first-order definable.*

Proof. By Isbell [25] and Werner [38] ([8, 4.3.4]), we have $\mathbf{M} := \langle \{0, 1\}; *, 0, 1, \leq \rangle$, where \leq is the usual order and $*$ interchanges 0 and 1. Define $\mathcal{M} := \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$. Starting from the Stralka space $\langle C; \leq_S, \mathcal{T} \rangle$, define the order $\leq := \leq_S \setminus \{(\frac{1}{3}, \frac{2}{3})\}$ on C , and so define the structure $\mathbf{X} := \langle C; *, 0, 1, \leq, \mathcal{T} \rangle$, where $x^* := 1 - x$. In Example 2.13 of [10], it is shown that $\mathbf{X} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$. Now we can apply the FUPT 5.2, since every minimal isolated substructure of \mathbf{X} is isomorphic to the substructure on $\{0, \frac{1}{3}, \frac{2}{3}, 1\}$ or $\{0, \frac{1}{9}, \frac{2}{9}, \frac{7}{9}, \frac{8}{9}, 1\}$. \square

Example 6.7. *The category $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{S})$, strongly dual to Stone algebras, is not first-order definable.*

Proof. By Davey [15, 16] ([8, 4.3.7]), we have $\mathbf{S} := \langle \{0, a, 1\}; d, \leq \rangle$, where \leq is the disconnected order with $1 \leq a$, and $d(0) = 0$, $d(a) = d(1) = 1$. Based on the Stralka space $\langle C; \leq_S, \mathcal{T} \rangle$, let \leq denote the order on C consisting of \leq_S together with all pairs $(0, x)$, for $x \in C$. Now define $\mathbf{X} := \langle C; d, \leq, \mathcal{T} \rangle$, where $d : C \rightarrow C$ is constant with value 0. In Example 2.9 of [10], it is shown that $\mathbf{X} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$, where $\mathcal{M} := \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{S})$. The result follows by the FUPT 5.2, as every minimal isolated substructure of \mathbf{X} is isomorphic to the substructure on $\{0, 1\}$ or $\{0, \frac{1}{3}, \frac{2}{3}\}$. \square

The proofs of the next two examples follow a similar line, being based on Examples 2.10 and 2.12 of [10], respectively.

Example 6.8. *The category $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{DS})$, strongly dual to double Stone algebras, is not first-order definable.*

Example 6.9. *The category $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{K})$, strongly dual to Kleene algebras, is not first-order definable.*

7. GRAPHS

In this section, we will examine the standardness of several classes of graphs. We shall see a striking demonstration of how a compatible Boolean topology places strong finiteness conditions on a graph. In Example 2.14, we saw that the uH-class of all graphs is standard. Here, we will apply the FILT 3.5, SILT 3.9 and SUPT 5.3 to give uH-classes of graphs that are not standard, not finitely axiomatizable, and that have a Boolean core that is not even first-order definable.

We consider a **graph** to be a structure $\mathbf{G} = \langle G; \sim \rangle$ that satisfies the uH-formula $x \sim y \implies y \sim x$. A graph is **simple** if satisfies the additional uH-formula $x \not\sim x$, and so has no loops. We will show that the uH-class of simple graphs is standard, and that this fact is related to graph colorings.

Fix a positive integer $k \in \mathbb{N}$. A **k -coloring** of a graph \mathbf{G} is a homomorphism from \mathbf{G} into the complete simple graph \mathbf{K}_k on the k vertices $\{0, 1, \dots, k-1\}$, which represent k colors. We say that \mathbf{G} is **k -colorable** if there exists a k -coloring of \mathbf{G} . So a graph is k -colorable if and only if it is ℓ -partite, for some $\ell \leq k$. Clearly, every k -colorable graph is simple.

The following example of a k -colorable graph will play an important role here. Define \mathbf{C}_k to be the graph with vertices $\{0, 1, \dots, k-1, k, k+1\}$ obtained from the complete simple graph \mathbf{K}_{k+2} by removing the three edges $k \sim 0 \sim k+1 \sim 1$. Then \mathbf{C}_k is k -colorable via $i \mapsto i \pmod{k}$.

Now define $\mathfrak{G}^{(k)}$ to be the class of all k -colorable graphs.

Proposition 7.1. (Nešetřil and Pultr [29]) *For each $k \in \mathbb{N}$, the class $\mathfrak{G}^{(k)}$ of all k -colorable graphs is the uH-class $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{C}_k)$.*

It follows that the class of all simple graphs is not generated as a universal Horn class by any finite collection of finite graphs. Indeed, such a collection must be k -colorable, for some $k \in \mathbb{N}$, and will therefore be contained in the uH-class $\mathfrak{G}^{(k)}$, which does not include \mathbf{K}_{k+1} .

Now consider a topological graph $\mathfrak{X} = \langle X; \sim, \mathcal{T} \rangle$. A **continuous k -coloring** of \mathfrak{X} is a continuous homomorphism from \mathfrak{X} into \mathbf{K}_k . So \mathfrak{X} is continuously k -colorable if and only if it is clopen ℓ -partite, for some $\ell \leq k$. (A compact graph with vertices in \mathbb{R}^2 is continuously finitely colorable if and only if it can be colored using a paint brush whose tip is an open disc of some positive radius.)

Proposition 7.2. *For each $k \in \mathbb{N}$, the class of all continuously k -colorable Boolean topological graphs is $\mathfrak{G}_{\text{Bc}}^{(k)} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{C}_k)$.*

Proof. Let \mathfrak{X} be a Boolean topological graph. First assume that $\mathfrak{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{C}_k)$. We can obtain a continuous k -coloring of \mathfrak{X} by composing any morphism from \mathfrak{X} to \mathbf{C}_k with a k -coloring of \mathbf{C}_k . Now assume that $\gamma : \mathfrak{X} \rightarrow \mathbf{K}_k$ is a continuous k -coloring. We will show that $\mathfrak{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{C}_k)$, via the Separation Theorem 1.3.

Let $a, b \in X$ with $a \neq b$. We want a continuous homomorphism $\varphi : \mathfrak{X} \rightarrow \mathbf{C}_k$ such that $\varphi(a) \neq \varphi(b)$. Since $\gamma : \mathfrak{X} \rightarrow \mathbf{K}_k \leq \mathbf{C}_k$ is a continuous homomorphism, we can assume that $\gamma(a) = \gamma(b)$. Without loss of generality, we can also assume that $\gamma(a) = 0$. Let U be a clopen subset of $\gamma^{-1}(0)$ containing a but not b . We can define the continuous homomorphism $\varphi : \mathfrak{X} \rightarrow \mathbf{C}_k$ by $\varphi(x) := k$, if $x \in U$, and $\varphi(x) := \gamma(x)$, otherwise. We have $\varphi(a) = k \neq 0 = \varphi(b)$, as desired.

Now let $a, b \in X$ with $a \approx b$. We want to find a $\varphi : \mathfrak{X} \rightarrow \mathbf{C}_k$ with $\varphi(a) \approx \varphi(b)$. Since \mathbf{C}_k has no loops, we can assume that $\gamma(a) \neq \gamma(b)$. Also assume that $\gamma(a) = 0$ and $\gamma(b) = 1$. Since \sim is closed in \mathfrak{X}^2 , there are clopen sets $U \subseteq \gamma^{-1}(0)$ and $V \subseteq \gamma^{-1}(1)$ such that $U \times V$ contains (a, b) but does not intersect \sim . Now we can define $\varphi : \mathfrak{X} \rightarrow \mathbf{C}_k$ by

$$\varphi(x) := \begin{cases} k & \text{if } x \in \gamma^{-1}(0) \setminus U, \\ k+1 & \text{if } x \in V, \\ \gamma(x) & \text{otherwise.} \end{cases}$$

We have $\varphi(a) = 0 \approx k+1 = \varphi(b)$, as required. \square

Define \mathfrak{G} to be the uH-class of all simple graphs, so that \mathfrak{G}_{Bt} is the class of all Boolean topological simple graphs.

Theorem 7.3.

- (i) *The uH-class \mathfrak{G} of all simple graphs is standard.*
- (ii) *Every topological graph in \mathfrak{G}_{Bt} is an inverse limit of finite simple graphs.*
- (iii) *Every topological graph in \mathfrak{G}_{Bt} is continuously k -colorable, for some $k \in \mathbb{N}$.*

Proof. We first prove (iii). Let $\mathfrak{X} = \langle X; \sim, \mathcal{T} \rangle$ be a Boolean topological simple graph. Then \sim is closed in \mathfrak{X}^2 . For each $a \in X$, we have $a \approx a$ and so there is a clopen subset U_a of \mathfrak{X} such that $a \in U_a$ and $(U_a \times U_a) \cap \sim = \emptyset$. As \mathfrak{X} is compact, there is a finite set $\mathcal{P} := \{U_1, U_2, \dots, U_k\} \subseteq \{U_a \mid a \in X\}$ covering X . By removing overlaps, we may assume that \mathcal{P} is a clopen partition of \mathfrak{X} . As $(U_i \times U_i) \cap \sim = \emptyset$, for each $i = 1, \dots, k$, we see that \mathcal{P} is the kernel of a continuous k -coloring of \mathfrak{X} .

To prove (i), let $\mathfrak{X} \in \mathfrak{G}_{\text{Bt}}$. By (iii), there is some $k \in \mathbb{N}$ such that \mathfrak{X} is continuously k -colorable. So we have $\mathfrak{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{C}_k) \subseteq \mathfrak{G}_{\text{Bc}}$, by Proposition 7.2. Thus \mathfrak{G} is standard, whence (i) holds. Part (ii) now follows from Theorem 2.6. \square

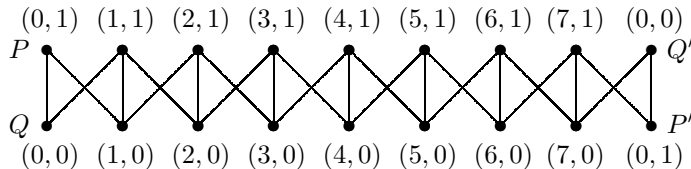


FIGURE 3. The graph \mathbf{X}_7 on a Möbius strip, for $k = 2$. (Glue PQ to $P'Q'$). This graph is not 2-colorable, but each 7-element subgraph is.

Combining Proposition 7.2 and Theorem 7.3, we see that the class \mathfrak{G}_{Bt} of Boolean topological simple graphs is the nested union of the classes of Boolean topological (continuously) k -colorable graphs, for all $k \in \mathbb{N}$:

$$\mathfrak{G}_{\text{Bt}} = \bigcup \{ \mathfrak{G}_{\text{Bc}}^{(k)} \mid k \in \mathbb{N} \} \subseteq \bigcup \{ \mathfrak{G}_{\text{Bt}}^{(k)} \mid k \in \mathbb{N} \} \subseteq \mathfrak{G}_{\text{Bt}}.$$

Whereas the uH-class \mathfrak{G} is standard, we will use the SILT 3.9 and SUPT 5.3 to show that, for each $k \geq 2$, the uH-class $\mathfrak{G}^{(k)}$ is neither standard nor finitely axiomatizable and the Boolean core $\mathfrak{G}_{\text{Bc}}^{(k)}$ is not first-order definable.

Throughout the rest of this section, we fix a positive integer $k \geq 2$ and consider the uH-class $\mathfrak{G}^{(k)}$ of k -colorable graphs. We shall construct an inverse system of finite simple graphs $\mathbf{X}_n = \langle X_n; \sim \rangle$, for $n \in \mathbb{N}$. (Roughly speaking, the graph \mathbf{X}_n will consist of $n+1$ copies of the complete graph \mathbf{K}_k linked together into a Möbius strip.) Define $X_n := \{0, 1, \dots, n\} \times \{0, 1, \dots, k-1\}$, and let \sim be the smallest symmetric relation on X_n satisfying

$$\begin{aligned} (i, a) &\sim (i, b) && \text{if } a \neq b, \\ (i, a) &\sim (i+1, b) && \text{if } a \neq b \text{ and } i < n, \\ (n, a) &\sim (0, b) && \text{if } a \not\equiv b+1 \pmod{k}, \end{aligned}$$

for all $i \leq n$ and $a, b < k$. Notice that, for all $i \leq n$, the subgraph $\mathbf{X}_{n,i}$ of \mathbf{X}_n on the set $\{i\} \times \{0, 1, \dots, k-1\}$ is a complete graph on k vertices. The following fact is easy to establish. (See Figure 3.)

Lemma 7.4. *Let $n \in \mathbb{N}$. The simple graph \mathbf{X}_n is not k -colorable. But, for all $i \leq n$, every subgraph of \mathbf{X}_n that does not intersect $\mathbf{X}_{n,i}$ is k -colorable. In particular, every subgraph of \mathbf{X}_n with at most n vertices is k -colorable.*

For $n \in \mathbb{N}$, we define a surjective connecting homomorphism $\varphi_n : \mathbf{X}_{n+1} \rightarrow \mathbf{X}_n$ by $\varphi_n(i, a) := (i, a)$, for $i \leq n$, and $\varphi_n(n+1, a) := (n, a)$. We define the inverse limit $\underline{\mathbf{X}} := \varprojlim \{ \mathbf{X}_n \mid n \in \mathbb{N} \}$. We can now apply the SILT 3.9 to the uH-class $\mathfrak{G}^{(k)}$. This class was originally shown to be not finitely axiomatizable by Taylor [36].

Theorem 7.5. *Let $k \geq 2$. The uH-class of all k -colorable graphs, $\mathfrak{G}^{(k)} = \mathbb{ISP}^+(\mathbf{C}_k)$, is neither standard nor finitely axiomatizable. In particular, there is a Boolean topological graph that is k -colorable but not continuously k -colorable.*

Proof. We apply the SILT 3.9 with $\underline{\mathbf{X}}$ as defined above. Condition S($\underline{\mathbf{X}}, \mathfrak{G}^{(k)}$) is given by Lemma 7.4. To verify pointwise non-separability, choose any $a \neq b$ in $\underline{\mathbf{X}}$. Now let $n \in \mathbb{N}$ and $\mathbf{M} \in \mathfrak{G}_{\text{fin}}^{(k)}$. The graph \mathbf{X}_n is not k -colorable, by Lemma 7.4, but \mathbf{M} is k -colorable. Thus there are no homomorphisms from \mathbf{X}_n into \mathbf{M} , and so $\underline{\mathbf{X}}$ is

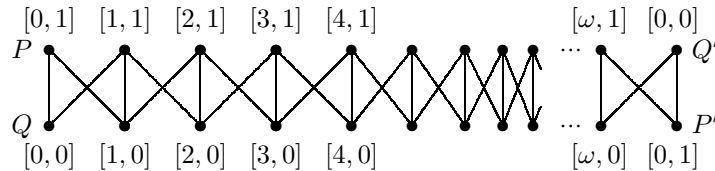


FIGURE 4. The topological graph \mathfrak{X} on a Möbius strip, for $k = 2$. (Glue PQ to $P'Q'$.) This graph is 2-colorable, but not continuously 2-colorable.

vacuously pointwise non-separable. By the SILT 3.9, it follows that $\mathfrak{X} \in \mathfrak{G}_{\text{Bt}}^{(k)} \setminus \mathfrak{G}_{\text{Bc}}^{(k)}$ and the class $\mathfrak{G}^{(k)}$ is not finitely axiomatizable. It now follows, by Proposition 7.2, that the topological graph \mathfrak{X} is k -colorable but not continuously k -colorable. \square

Figures 3 and 4 illustrate these ideas for the case $k = 2$. We can try to move around the graph \mathbf{X}_7 , coloring it with two colors, but we will reach an impasse when we arrive back at the start. If we do the same with the graph \mathfrak{X} , we will be successful; there is no impasse, because there is no connection to $[\omega, 0]$ or $[\omega, 1]$ from the left. But coloring the vertices of \mathfrak{X} will require an infinite collection of arbitrarily small paint brushes. In contrast, continuously coloring \mathfrak{X} requires that we use a single brush of some finite size. This more physically realistic task will fail, exactly as the coloring the finite graph \mathbf{X}_7 failed.

The previous example illustrates the advantage of our inverse-limit techniques. We did not need to work directly with the infinite Boolean topological graph \mathfrak{X} in the proof of Theorem 7.5. We only needed to work with finite graphs to reach a topological conclusion.

Theorem 7.6. *Let $k \geq 2$. The class of all continuously k -colorable Boolean topological graphs, $\mathfrak{G}_{\text{Bc}}^{(k)} = \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{C}_k)$, is not first-order definable.*

Proof. We apply the SUPT 5.3 to the topological graph $\mathfrak{X} \in \mathfrak{G}_{\text{Bt}}^{(k)} \setminus \mathfrak{G}_{\text{Bc}}^{(k)}$ from the previous proof. Clearly, every graph is locally finite. As illustrated in Figure 4, the vertices of \mathfrak{X} are

$$\begin{aligned}
 [i, a] &:= ((1, a), \dots, (i-1, a), (i, a), (i, a), \dots) && \text{for } i < \omega \text{ and } a < k, \text{ and} \\
 [\omega, a] &:= ((1, a), (2, a), (3, a), (4, a), (5, a), \dots) && \text{for } a < k,
 \end{aligned}$$

and \mathfrak{X} has a k -space topology, with the k -compactification $f : X \rightarrow X$ given by $f([i, a]) := [\omega, a]$, for all $i \leq \omega$. Now let $\langle \mathbf{Y}; g \rangle$ be a model of the universal theory of $\langle \mathbf{X}; f \rangle$. It remains to verify that the edge relation \sim of \mathbf{Y} is closed in Y^2 under the topology \mathcal{T}^g .

We use the following four statements about $\langle \mathbf{Y}; g \rangle$, which hold because they are expressible by quantifier-free formulæ that hold universally in $\langle \mathbf{X}; f \rangle$:

- (1) \mathbf{Y} is a simple graph;
- (2) each pair of distinct vertices of $g(Y)$ is adjacent;
- (3) for all $x \in g(Y)$, the subgraph of \mathbf{Y} on the set $g^{-1}(x)$ has at most one edge;
- (4) each vertex of \mathbf{Y} is adjacent to at most $3(k-1)$ other vertices.

To see that \sim is closed, let $x, y \in Y$ with $x \approx y$. There are three cases.

Case 1: $x, y \notin g(Y)$. The sets $\{x\}$ and $\{y\}$ are open in \mathcal{T}^g , and $\{x\} \times \{y\}$ does not intersect \sim .

Case 2: $x, y \in g(Y)$. We have $x = y$, by (2). By (1) and (3), there is a cofinite subset W of $g^{-1}(x)$ that contains x and, as a subgraph of \mathbf{Y} , has no edges. The set W is open in \mathcal{T}^g , and W^2 contains $(x, y) = (x, x)$ but does not intersect \sim .

Case 3: $x \in g(Y)$ and $y \notin g(Y)$. By (4), the set U of all vertices of \mathbf{Y} that are not adjacent to y is cofinite in Y . Both U and $\{y\}$ are open in \mathcal{T}^g , and the set $U \times \{y\}$ contains (x, y) but does not intersect \sim . \square

We close this section by considering the uH-class \mathcal{D} of **dotted graphs**, defined by the uH-formulae $x \sim y \implies y \sim x$ and $(x \sim x) \wedge (x \sim y) \implies x \approx y$. A dotted graph is a disjoint union of a simple graph and a set of disconnected looped vertices. In contrast with the uH-class of simple graphs, which is standard, we shall find that \mathcal{D} is non-standard and that \mathcal{D}_{Bc} is not even first-order definable.

We establish the following result with the FILT 3.5 and the SUPT 5.3. Since \mathcal{D} is finitely axiomatizable, the SILT 3.9 is not applicable. But Theorem 3.14 assures us that we could use the TILT 3.10. Recall that \mathbf{C}_2 is the path of length 3 and that, in this context, $\mathbf{1}$ is the one-element looped graph.

Proposition 7.7. *Let \mathcal{M} be a uH-class of dotted graphs with $\mathbf{C}_2 \dot{\cup} \mathbf{1} \in \mathcal{M}$. Then \mathcal{M} is not even pre-standard, and \mathcal{M}_{Bc} is not even first-order definable.*

Proof. We will apply the FILT 3.5. Define the graph $\mathbf{X}_n := \langle \{0, 1, \dots, n\}; \sim \rangle$ to be a path of length n with a loop at the end: $0 \sim 1 \sim \dots \sim n-1 \sim n \sim n$. Define the surjective homomorphism $\varphi_n : \mathbf{X}_{n+1} \rightarrow \mathbf{X}_n$ by $\varphi_n(i) := i$, for $i \leq n$, and $\varphi_n(n+1) := n$. Now define $\mathbf{X} := \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$, every homomorphism from \mathbf{X}_n to a dotted graph is constant. Thus the pointwise non-separability of \mathbf{X} with respect to \mathcal{M} is witnessed by any $a \neq b$ in X . We will verify $F(\mathbf{X}, \mathbb{ISP}^+(\mathbf{C}_2 \dot{\cup} \mathbf{1}))$. Let $n \in \mathbb{N}$ and assume that φ_n is one-to-one on $Y \subseteq X_{n+1}$. Then $n \notin Y$ or $n+1 \notin Y$. Since $\mathbb{ISP}^+(\mathbf{C}_2)$ consists of all 2-colorable graphs, we conclude that $\text{Sg}_{\mathbf{X}_{n+1}}(Y) \in \mathbb{ISP}^+(\mathbf{C}_2 \dot{\cup} \mathbf{1}) \subseteq \mathcal{M}$. By the FILT 3.5, we have $\mathbf{X} \notin \mathcal{M}_{\text{Bc}}$ and $\mathbf{X} \in [\mathbb{ISP}^+(\mathbf{C}_2 \dot{\cup} \mathbf{1})]_{\text{Bt}} \subseteq [\mathbb{ISP}^+(\mathcal{M}_{\text{fin}})]_{\text{Bt}}$. So \mathcal{M} is not even pre-standard.

It is easy to see that the inverse limit \mathbf{X} is isomorphic to the Boolean topological graph $\mathbf{Z} := \langle \{0, 1, 2, \dots\} \cup \{\omega\}; \sim, \mathcal{T} \rangle$, where

- the graph \mathbf{Z} is the disjoint union of a semi-infinite path and a looped vertex, $0 \sim 1 \sim 2 \sim \dots$ and $\omega \sim \omega$, and
- the topology on \mathbf{Z} comes from the 1-compactification $f : Z \rightarrow \{\omega\}$.

We can now show that \mathcal{M}_{Bc} is not first-order definable, by verifying the conditions of the SUPT 5.3. Note that the following properties of $\langle \mathbf{Z}; f \rangle$ can be expressed via universal sentences: every vertex is adjacent to at most two other vertices; the compactification point ω is the unique looped vertex. \square

8. UNARY ALGEBRAS

An algebra is **unary** if each of its operations is unary; it is a **unar** if, in addition, it has only one operation. A unary algebra can be particularly tractable, because it can be specified by a multi-digraph, and because a union of subalgebras is a subalgebra. Natural dualities whose dual category is a unary topological quasivariety are of particular value because of their tractability, and are studied in detail in Chapter 6 of [8].

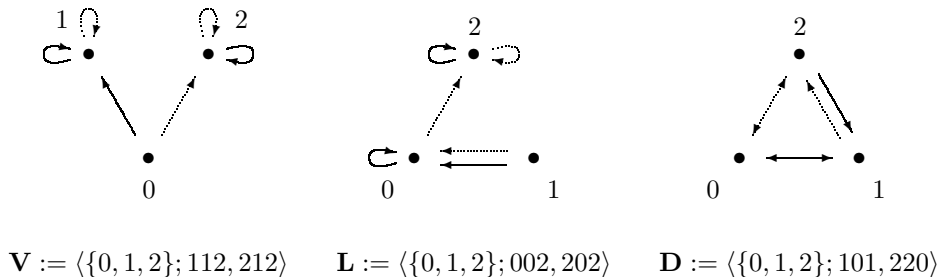


FIGURE 5

Unary algebras can provide a valuable testing ground for gauging the difficulty of an algebraic problem. If a problem reveals a complex solution for a simple class of unary algebras, it may be fair to conclude that it is likely to be a difficult problem in general. For example, Bestsennyĭ [4] examined the class of 3-element unary algebras and found, even there, that the description of the finitely uH-based ones (which we give below) is quite non-trivial. Similarly, Clark, Davey and Pitkethly [12] showed that the dualisability problem for 3-element unary algebras is surprisingly difficult.

Every finite unar is finitely uH-based and standard [10], whereas Example 2.16 shows that the class of all unars is non-standard. In this section, we use the SILT 3.9 and SUPT 5.3 to show that every 3-element unary algebra that is not finitely uH-based is also non-standard and generates a topological prevariety that is not first-order definable. We also show that, for every finite unary algebra \mathbf{U} with more than one operation, there are finite unary algebras \mathbf{A} and \mathbf{B} containing \mathbf{U} such that \mathbf{A} is finitely uH-based and standard and \mathbf{B} is not finitely uH-based and non-standard.

For $u_0, u_1, u_2 \in \{0, 1, 2\}$, we let the string $u_0u_1u_2$ denote the unary operation on $\{0, 1, 2\}$ given by $i \mapsto u_i$.

Theorem 8.1. (Bestsennyĭ [4]) *Let \mathbf{U} be a 3-element unary algebra of finite type. Then \mathbf{U} is finitely uH-based if and only if it does not have a term reduct isomorphic to one of the three algebras in Figure 5.*

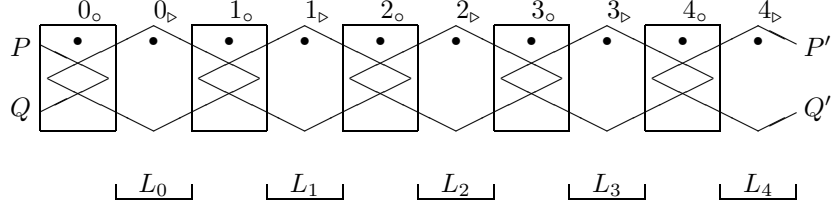
Let us now fix a unary algebra \mathbf{U} of finite type on the set $\{0, 1, 2\}$. We are aiming to apply the SILT 3.9 to show that, if \mathbf{U} is not finitely uH-based, then $\text{ISP}^+(\mathbf{U})$ is not standard. We start by constructing a particular subalgebra of \mathbf{U}^ω . (Our construction is a special case of one introduced by Pitkethly [32].) For each $i \in \omega$, define the two sequences i_\circ and i_\triangleright in U^ω by

$$i_\circ(m) = \begin{cases} 2 & \text{if } m \leq i, \\ 1 & \text{if } m > i, \end{cases} \quad \text{and} \quad i_\triangleright(m) = \begin{cases} 2 & \text{if } m \leq i, \\ 0 & \text{if } m = i + 1, \\ 1 & \text{if } m > i + 1. \end{cases}$$

Now define the unary algebra

$$\mathbf{Z} := \text{Sg}_{\mathbf{U}^\omega} (\{i_\circ \mid 0 \leq i < \omega\} \cup \{i_\triangleright \mid 0 \leq i < \omega\}),$$

and define C to be the set of all constant sequences in U^ω . We use two basic lemmas about \mathbf{Z} . The first says that \mathbf{Z} has a high degree of uniformity.

FIGURE 6. Non-constant elements of \mathbf{X}_4 . (Glue PQ to $P'Q'$.)**Lemma 8.2.**

- (i) *There is an embedding $\xi : \mathbf{Z} \rightarrow \mathbf{Z}$ that maps i_\circ to $(i+1)_\circ$ and maps i_\triangleright to $(i+1)_\triangleright$, for all $i \in \omega$, and fixes each element of $Z \cap C$.*
- (ii) *There is a homomorphism $f : \mathbf{Z} \rightarrow \mathbf{Z}$ that maps i_\circ and i_\triangleright to 0_\circ , for all $i \in \omega$.*

Proof. For (i), note that $\xi(z)(0) = z(0)$ and $\xi(z)(m) = z(m-1)$, for $z \in Z$ and $m \geq 1$. For (ii), choose generators $x, y \in \{i_\circ \mid i \in \omega\} \cup \{i_\triangleright \mid i \in \omega\}$. Let s and t be unary term functions of \mathbf{U} , and assume that $s^{\mathbf{Z}}(x) = t^{\mathbf{Z}}(y)$. We have $x(0) = y(0) = 2$, and there is some $m \in \omega$ such that $x(m) = y(m) = 1$. Thus

$$s(2) = s^{\mathbf{Z}}(x)(0) = t^{\mathbf{Z}}(y)(0) = t(2) \quad \text{and} \quad s(1) = s^{\mathbf{Z}}(x)(m) = t^{\mathbf{Z}}(y)(m) = t(1).$$

So $s^{\mathbf{Z}}(0_\circ) = t^{\mathbf{Z}}(0_\circ)$, and therefore $s^{\mathbf{Z}}(f(x)) = s^{\mathbf{Z}}(0_\circ) = t^{\mathbf{Z}}(0_\circ) = t^{\mathbf{Z}}(f(y))$. Thus f is a well-defined homomorphism. \square

The second basic lemma says that the 1-generated subalgebras of \mathbf{Z} have very limited intersections. The details are easy to check by writing out the generators.

Lemma 8.3. *Let $i, j < \omega$ with $i \neq j$. Then*

- (i) $\text{Sg}_{\mathbf{Z}}(i_\circ) \cap \text{Sg}_{\mathbf{Z}}(j_\circ) = Z \cap C$,
- (ii) $\text{Sg}_{\mathbf{Z}}(i_\triangleright) \cap \text{Sg}_{\mathbf{Z}}(j_\triangleright) \subseteq C$, if $i+1 < j$,
- (iii) $\text{Sg}_{\mathbf{Z}}(i_\triangleright) \cap \text{Sg}_{\mathbf{Z}}(j_\triangleright) \subseteq \text{Sg}_{\mathbf{Z}}(j_\circ)$, if $i+1 = j$, and
- (iv) $\text{Sg}_{\mathbf{Z}}(i_\triangleright) \cap \text{Sg}_{\mathbf{Z}}(j_\circ) \subseteq C$, if $j \notin \{i, i+1\}$.

Let $n \in \mathbb{N} \cup \{0\}$ and define a subalgebra of \mathbf{Z} by

$$\mathbf{Z}_n := \text{Sg}_{\mathbf{Z}}(\{i_\circ \mid 0 \leq i \leq n+1\} \cup \{i_\triangleright \mid 0 \leq i \leq n\}).$$

We shall next define a quotient \mathbf{X}_n of \mathbf{Z}_n . Using the previous two lemmas, we obtain an isomorphism $\psi_n : \text{Sg}_{\mathbf{Z}}((n+1)_\circ) \rightarrow \text{Sg}_{\mathbf{Z}}(0_\circ)$ that maps $(n+1)_\circ$ to 0_\circ and fixes $Z \cap C = \text{Sg}_{\mathbf{Z}}(0_\circ) \cap \text{Sg}_{\mathbf{Z}}((n+1)_\circ)$. Thus we can define \mathbf{X}_n to be the quotient of \mathbf{Z}_n obtained by identifying $\text{Sg}_{\mathbf{Z}_n}((n+1)_\circ)$ with $\text{Sg}_{\mathbf{Z}_n}(0_\circ)$ using ψ_n . For notational convenience, we assume that the elements of \mathbf{X}_n coincide with those of $(\mathbf{Z}_n \setminus \text{Sg}_{\mathbf{Z}_n}((n+1)_\circ)) \cup (Z \cap C)$.

Figure 6 effectively summarizes the information given in Lemma 8.3, which tells us how the 1-generated subalgebras of \mathbf{Z}_n and \mathbf{X}_n might possibly intersect outside of C . (The unary algebra \mathbf{X}_n can be thought of as a bracelet made up of $n+1$ beads (the rectangles) with $n+1$ links between them.) The embedding ξ from Lemma 8.2 induces an automorphism ξ_n of \mathbf{X}_n , where i_\circ and i_\triangleright map to $(i \oplus_{n+1} 1)_\circ$ and $(i \oplus_{n+1} 1)_\triangleright$, respectively.

We now want to create an inverse system, by defining a surjective connecting homomorphism $\varphi_n : \mathbf{X}_{n+1} \rightarrow \mathbf{X}_n$, for each $n \in \mathbb{N}$. We wish to define φ_n on the

generators of \mathbf{X}_{n+1} as follows:

$$\varphi_n(i_\circ) := \begin{cases} i_\circ & \text{if } i \leq n, \\ 0_\circ & \text{if } i = n + 1, \end{cases} \quad \text{and} \quad \varphi_n(i_\triangleright) := \begin{cases} i_\triangleright & \text{if } i \leq n, \\ 0_\circ & \text{if } i = n + 1, \end{cases}$$

for all $0 \leq i \leq n + 1$. (Less formally, we wish to turn the $(n + 2)$ -bracelet \mathbf{X}_{n+1} into an $(n + 1)$ -bracelet by collapsing bead $n + 1$ and link $n + 1$ onto bead 0 .)

Lemma 8.4. *The homomorphism $\varphi_n : \mathbf{X}_{n+1} \rightarrow \mathbf{X}_n$ is well defined, for each $n \in \mathbb{N}$.*

Proof. The algebra \mathbf{X}_{n+1} has \mathbf{Z}_n as a subalgebra, and $\varphi_n \upharpoonright_{\mathbf{Z}_n} : \mathbf{Z}_n \rightarrow \mathbf{X}_n$ is easily seen to be the natural homomorphism used to define \mathbf{X}_n . Now define $\mathbf{W} := \text{Sg}_{\mathbf{X}_{n+1}}(\{(n + 1)_\circ, (n + 1)_\triangleright, 0_\circ\})$. Then Lemma 8.2 can be used to deduce that $\varphi_n \upharpoonright_{\mathbf{W}} : \mathbf{W} \rightarrow \text{Sg}_{\mathbf{X}_n}(0_\circ)$ is well defined. As $Z_n \cap W = \text{Sg}_{\mathbf{X}_{n+1}}(\{(n + 1)_\circ, 0_\circ\})$, it follows that φ_n is a well-defined homomorphism. \square

We can now define the surjective inverse limit $\mathbf{X} := \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$. It turns out that condition S($\mathbf{X}, \mathbb{ISP}^+(\mathbf{U})$) of the SILT 3.9 holds for any choice of the 3-element unary algebra \mathbf{U} .

Lemma 8.5. *Let $n \in \mathbb{N}$ and let \mathbf{Y} be a subalgebra of \mathbf{X}_n that is generated by at most n elements. Then $\mathbf{Y} \in \mathbb{ISP}^+(\mathbf{U})$.*

Proof. Let S be a generating set for \mathbf{Y} with $|S| \leq n$. For each $0 \leq j \leq n$, define

$$\mathbf{W}_j := \text{Sg}_{\mathbf{X}_n}(\{i_\circ \mid 0 \leq i \leq n\} \cup \{i_\triangleright \mid 0 \leq i \leq n \text{ and } i \neq j\})$$

and $L_j := X_n \setminus W_j \subseteq \text{Sg}_{\mathbf{X}_n}(j_\triangleright)$; see Figure 6. Then we have $\mathbf{W}_n = \mathbf{Z}_{n-1}$. Using the automorphism ξ_n of \mathbf{X}_n (induced by ξ from Lemma 8.2), it follows that $\mathbf{W}_j \cong \mathbf{Z}_{n-1}$, for all $0 \leq j \leq n$. By Lemma 8.3, the subsets L_0, L_1, \dots, L_n of X_n are pairwise disjoint. Thus there exists $k \leq n$ such that $S \subseteq X_n \setminus L_k = W_k$, and consequently $\mathbf{Y} \leq \mathbf{W}_k \cong \mathbf{Z}_{n-1} \leq \mathbf{U}^\omega$. Hence $\mathbf{Y} \in \mathbb{ISP}^+(\mathbf{U})$. \square

It is the pointwise non-separability of \mathbf{X} , needed to apply the SILT 3.9, that requires the assumption that the 3-element unary algebra \mathbf{U} is not finitely uH-based. Under this assumption, the 1-generated subalgebras of \mathbf{X}_n actually do overlap enough to connect \mathbf{X}_n into the bracelet shown in Figure 6.

We use the description from Theorem 8.1 of the 3-element unary algebras that are not finitely uH-based.

Lemma 8.6. *Assume that \mathbf{U} has \mathbf{V} or \mathbf{L} as a term reduct. Then, for all $n \in \mathbb{N}$, every homomorphism $\psi : \mathbf{X}_n \rightarrow \mathbf{U}$ satisfies $\psi(0_\circ) = \psi(1_\triangleright)$.*

Proof. There are distinct $p, q \in \{0, 1, 2\}$ such that ppq and qpq are term functions of \mathbf{U} . In the algebra $\mathbf{Z} \leq \mathbf{U}^\omega$, we have $qpq(i_\triangleright) = ppq((i + 1)_\triangleright)$, for all $i < \omega$. It follows that \mathbf{X}_n satisfies $qpq(i_\triangleright) = ppq(j_\triangleright)$, for $0 \leq i < n + 1$ and $j := i \oplus_{n+1} 1$, and so we have

$$(\spadesuit) \quad qpq(\psi(i_\triangleright)) = \psi(qpq(i_\triangleright)) = \psi(ppq(j_\triangleright)) = ppq(\psi(j_\triangleright)).$$

Using \spadesuit , we now have

$$(\diamond) \quad \psi(i_\triangleright) = 2 \implies ppq(\psi(j_\triangleright)) = qpq(\psi(i_\triangleright)) = q \implies \psi(j_\triangleright) = 2$$

and

$$(\heartsuit) \quad \psi(j_\triangleright) = 1 \implies qpq(\psi(i_\triangleright)) = ppq(\psi(j_\triangleright)) = p \implies \psi(i_\triangleright) = 1,$$

for all $0 \leq i < n + 1$ and $j := i \oplus_{n+1} 1$.

Since the addition is modulo $n+1$, we can apply \diamond and \heartsuit repeatedly to conclude that $\psi(0_{\triangleright}) = 2 \iff \psi(1_{\triangleright}) = 2$ and $\psi(0_{\triangleright}) = 1 \iff \psi(1_{\triangleright}) = 1$. Because the co-domain of ψ is $\{0, 1, 2\}$, it follows that $\psi(0_{\triangleright}) = \psi(1_{\triangleright})$. \square

Unfortunately, our construction does not work so directly in the case that the unary algebra $\mathbf{U} = \langle \{0, 1, 2\}; F \rangle$ has \mathbf{D} as a term reduct. We define the unary algebra $\mathbf{U}^{\sharp} := \langle \{0, 1, 2\}; F \cup \{021\} \rangle$, by adding the unary operation 021 to \mathbf{U} . Now we can construct the algebras \mathbf{Z}^{\sharp} , \mathbf{Z}_n^{\sharp} and \mathbf{X}_n^{\sharp} of type $F \cup \{021\}$ using \mathbf{U}^{\sharp} , as described above. Define \mathbf{Z}^b , \mathbf{Z}_n^b and \mathbf{X}_n^b to be the reducts of \mathbf{Z}^{\sharp} , \mathbf{Z}_n^{\sharp} and \mathbf{X}_n^{\sharp} of type F .

Lemma 8.7. *Assume that \mathbf{U} has \mathbf{D} as a term reduct. Then, for all $n \in \mathbb{N}$, every homomorphism $\psi : \mathbf{X}_n^b \rightarrow \mathbf{U}$ satisfies $\psi(0_{\triangleright}) = \psi(1_{\triangleright})$.*

Proof. The unary operations 101 and 220 on $\{0, 1, 2\}$ are term functions of \mathbf{U} . So $010 = 101 \circ 101$ and $002 = 220 \circ 220$ are also term functions of \mathbf{U} . The algebra \mathbf{X}_n^b is closed under 021, although it is not necessarily a term function.

Let $0 \leq i < n+1$ and define $j := i \oplus_{n+1} 1$. Using the definition of \mathbf{X}_n^{\sharp} from $\mathbf{Z}_n^{\sharp} \leq (\mathbf{U}^{\sharp})^{\omega}$, we can check that \mathbf{X}_n^b satisfies

$$(\spadesuit) \quad 101(i_{\triangleright}) = 010 \circ 021(j_{\circ}) \quad \text{and} \quad (\clubsuit) \quad 220(j_{\triangleright}) = 002 \circ 021(j_{\circ}).$$

We next want to prove that

$$(\diamond) \quad \psi(i_{\triangleright}) = 2 \implies \psi(j_{\triangleright}) = 2 \quad \text{and} \quad (\heartsuit) \quad \psi(j_{\triangleright}) = 1 \implies \psi(i_{\triangleright}) = 1.$$

To prove \diamond , assume that $\psi(i_{\triangleright}) = 2$. Then, using \spadesuit , we have

$$\begin{aligned} 101(\psi(021(j_{\circ}))) &= 101 \circ 010(\psi(021(j_{\circ}))) = 101(\psi(010 \circ 021(j_{\circ}))) \\ &= 101(\psi(101(i_{\triangleright}))) = 010(\psi(i_{\triangleright})) = 010(2) = 0, \end{aligned}$$

and so $\psi(021(j_{\circ})) = 1$. Thus \clubsuit gives us

$$220(\psi(j_{\triangleright})) = \psi(220(j_{\triangleright})) = \psi(002 \circ 021(j_{\circ})) = 002(\psi(021(j_{\circ}))) = 002(1) = 0,$$

whence $\psi(j_{\triangleright}) = 2$. Thus \diamond holds.

The proof of \heartsuit is similar. Assume that $\psi(j_{\triangleright}) = 1$. Then, using \clubsuit , we have $220(\psi(021(j_{\circ}))) = 220(\psi(220(j_{\triangleright}))) = 002(1) = 0$, whence $\psi(021(j_{\circ})) = 2$. Thus \spadesuit gives us $101(\psi(i_{\triangleright})) = \psi(010 \circ 021(j_{\circ})) = 010(2) = 0$. So $\psi(i_{\triangleright}) = 1$ and \heartsuit holds.

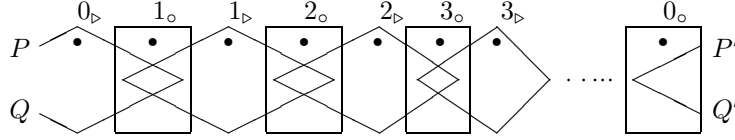
It is now easy to use \diamond and \heartsuit to show that $\psi(0_{\triangleright}) = \psi(1_{\triangleright})$. \square

Theorem 8.8. *Let \mathbf{U} be a 3-element unary algebra of finite type. If \mathbf{U} is not finitely uH-based, then \mathbf{U} is non-standard.*

Proof. By Theorem 8.1, we can consider two cases. First assume that \mathbf{V} or \mathbf{L} is a term reduct of \mathbf{U} . Lemmas 8.4, 8.5 and 8.6 show that the SILT 3.9 applies to the surjective inverse limit $\mathfrak{X} := \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$. Thus \mathbf{U} is non-standard.

Now assume that \mathbf{D} is a term reduct of \mathbf{U} . Lemmas 8.4 and 8.5 apply to \mathbf{X}_n^b and \mathbf{U} , as they apply to \mathbf{X}_n^{\sharp} and \mathbf{U}^{\sharp} . So, also using Lemma 8.7, we can apply the SILT 3.9 to the inverse limit $\mathfrak{X} := \varprojlim \{\mathbf{X}_n^b \mid n \in \mathbb{N}\}$. Thus \mathbf{U} is non-standard. \square

We would like to strengthen Theorem 8.8 by applying the SUPT 5.3 to show that $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{U})$ is not first-order definable. To this end, we need to take a closer look at $\mathfrak{X} := \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$. We constructed \mathfrak{X} as an inverse limit in order to use the SILT 3.9. In fact, we can obtain \mathfrak{X} by simply adding a topology to the subalgebra

FIGURE 7. Non-constant elements of \mathcal{Z} . (Glue PQ to $P'Q'$.)

\mathbf{Z} of \mathbf{U}^ω . By Lemma 8.2, there is a retraction $f : \mathbf{Z} \rightarrow \text{Sg}_{\mathbf{Z}}(0_o)$. We define \mathcal{Z} by adding the λ -space topology \mathcal{T}^f to \mathbf{Z} , where $\lambda := |\text{Sg}_{\mathbf{Z}}(0_o)|$; see Figure 7.

Lemma 8.9. *The inverse limit $\mathcal{X} := \varprojlim \{\mathbf{X}_n \mid n \in \mathbb{N}\}$ is isomorphic to \mathcal{Z} .*

Proof. First note that the generators of the inverse limit \mathcal{X} are

$$[i_o] := (\overbrace{0_o, \dots, 0_o}^{i-1 \text{ times}}, i_o, i_o, \dots) \quad \text{and} \quad [i_p] := (\overbrace{0_o, \dots, 0_o}^{i-1 \text{ times}}, i_p, i_p, \dots),$$

for all $i < \omega$. For each $n \in \mathbb{N}$, we can define a continuous surjection $f_n : \mathcal{Z} \rightarrow \mathbf{X}_n$ by $f_n := \nu_n \cup f|_{\mathcal{Z} \setminus \mathbf{Z}_n}$, where $\nu_n : \mathbf{Z}_n \rightarrow \mathbf{X}_n$ is the natural map. (In other words, the map f_n collapses every bead and link from $n+1$ onwards onto bead 0.) The product map $\prod \{f_n \mid n \in \mathbb{N}\} : \mathcal{Z} \rightarrow \mathcal{X}$ is easily seen to be an isomorphism. \square

Theorem 8.10. *Let \mathbf{U} be a 3-element unary algebra of finite type. If \mathbf{U} is not finitely uH-based, then $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{U})$ is not first-order definable.*

Proof. Using Theorem 8.1, we may assume that \mathbf{U} has \mathbf{V} , \mathbf{L} or \mathbf{D} as a reduct. Define $\mathcal{M} := \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{U})$ and define the topological unary algebra \mathcal{Z} as above. (In the case that \mathbf{U} has \mathbf{D} as a reduct, we must first define \mathcal{Z}^\sharp from \mathbf{U}^\sharp , and then define \mathcal{Z} to be the reduct of \mathcal{Z}^\sharp in \mathcal{M}_{Bt} .) By Lemma 8.9 and the proof of Theorem 8.8, we have $\mathcal{Z} \in \mathcal{M}_{\text{Bt}} \setminus \mathcal{M}_{\text{Bc}}$. We will verify the conditions of the SUPT 5.3 for \mathcal{Z} , which has the λ -compactification $f : \mathbf{Z} \rightarrow \mathbf{Z}$.

Clearly, \mathbf{Z} is locally finite. Now let $\langle \mathbf{Y}; g \rangle$ be a model of the universal theory of $\langle \mathbf{Z}; f \rangle$. Then $g : \mathbf{Y} \rightarrow \mathbf{Y}$ is a λ -compactification of Y . We must show that each unary term function τ of \mathbf{Y} is continuous with respect to \mathcal{T}^g .

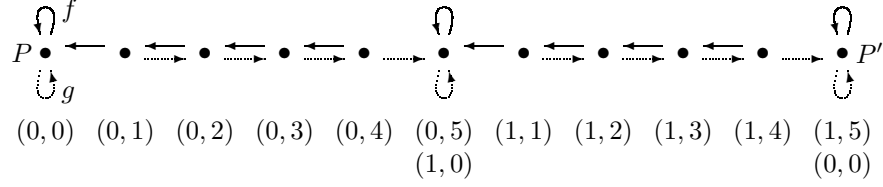
Let S be an open subset of Y under the topology \mathcal{T}^g . To prove that $T := \tau^{-1}(S)$ is also open, let $a \in T \cap g(Y)$. We now only need to show that $g^{-1}(a) \setminus T$ is finite. The set $g(Y)$ is a subuniverse of \mathbf{Y} , and so $\tau(a) \in S \cap g(Y)$. Since S is open, we know that $W := g^{-1}(\tau(a)) \setminus S$ is a finite subset of $Y \setminus g(Y)$.

By Lemmas 8.2 and 8.3 (see Figure 7), there is a $k \in \mathbb{N}$ such that, for each $b \in \mathcal{Z} \setminus f(\mathcal{Z})$, we have $|\tau^{-1}(b)| \leq k$. Since this is expressible via a quantifier-free formula, the same is true of \mathbf{Y} , and so the set $\tau^{-1}(W)$ is finite. But

$$\begin{aligned} g^{-1}(a) \setminus T &= \{y \in Y \mid g(y) = a \text{ and } \tau(y) \notin S\} \\ &\subseteq \{y \in Y \mid g \circ \tau(y) = \tau(a) \text{ and } \tau(y) \notin S\} = \tau^{-1}(W), \end{aligned}$$

as $g : \mathbf{Y} \rightarrow \mathbf{Y}$ is a homomorphism. Thus $g^{-1}(a) \setminus T$ is finite, as required. \square

As mentioned earlier, every finite unar is standard and finitely uH-based. To complete this section, we show that every finite unary algebra with more than one operation embeds into a finite algebra that is standard and finitely uH-based, and also embeds into a finite algebra that is not standard and not finitely uH-based.

FIGURE 8. The unary algebra \mathbf{X}_1 , for $k = 4$. (Glue P to P' .)

Lemma 8.11. *Let \mathbf{U} be a finite unary algebra of finite type. Then \mathbf{U} embeds into a finite unary algebra \mathbf{W} that is finitely uH-based and standard.*

Proof. This is basically Proposition 5.8 of [9]. We can assume that \mathbf{U} is non-empty. Let \mathbf{U}' be the unary algebra obtained from \mathbf{U} by adding an isolated fixed point. Then \mathbf{U}' has no constant term functions, and it follows that the equational class \mathcal{V} generated by \mathbf{U}' is closed under taking disjoint unions. Up to isomorphism, the variety \mathcal{V} has finitely many subdirectly irreducible algebras, each one finite (Higgs [24]). Let $\mathbf{A}_1, \dots, \mathbf{A}_n$ be a complete list of non-isomorphic subdirectly irreducibles in \mathcal{V} . Define \mathbf{W} to be the disjoint union of the algebras $\mathbf{U}', \mathbf{A}_1, \dots, \mathbf{A}_n \in \mathcal{V}$, so that $\mathbf{W} \in \mathcal{V}$ and $\mathbf{U} \leq \mathbf{W}$. We have $\mathbb{ISP}^+(\mathbf{W}) = \mathcal{V}$. Thus \mathbf{W} is standard, by Theorem 2.13, because the variety generated by any finite unary algebra has FDSC; see [9]. Also, the algebra \mathbf{W} is finitely uH-based, because the uH-class that it generates is a variety, and the variety generated by any finite unary algebra of finite type is finitely based. \square

Essentially the same proof shows that, for any finite set \mathcal{K} of finite unary algebras of finite type, there exists a finite unary algebra \mathbf{W} that is standard and finitely uH-based and satisfies $\mathcal{K} \subseteq \mathbb{ISP}^+(\mathbf{W})$. Now we prove the other embedding.

Lemma 8.12. *Let \mathbf{U} be a finite unary algebra with at least two operations. Then \mathbf{U} embeds into a finite unary algebra \mathbf{W} that is not finitely uH-based and non-standard.*

Proof. Assume that the unary algebra \mathbf{U} is non-empty and has type F , where $|F| \geq 2$. Choose any two distinct operations f and g from F . Define $k := |U|$ and define the unary algebra $\mathbf{Y} := \langle \{0, 1, \dots, k+1\}; F \rangle$ such that

- each operation in $F \setminus \{f, g\}$ is the identity on \mathbf{Y} ,
- the elements 0 and $k+1$ are fixed by f and g , and
- we have $f(i) := i-1$ and $g(i) := i+1$, for all $i \in Y \setminus \{0, k+1\}$.

It is easy to check that the only non-trivial homomorphic images of \mathbf{Y} have size at least $k+1 > |U|$. So every homomorphism from \mathbf{Y} to \mathbf{U} is constant.

Let \mathbf{W} denote the disjoint union of \mathbf{U} and \mathbf{Y} . We will apply the SILT 3.9 to $\mathbb{ISP}^+(\mathbf{W})$. For each $n \in \mathbb{N}$, we define \mathbf{Y}_n to be a disjoint union of $n+1$ copies of \mathbf{Y} , with the universe of \mathbf{Y}_n being $\{0, 1, \dots, n\} \times Y$. Now define \mathbf{X}_n by identifying the fixed points $(i, k+1)$ and $(i \oplus_{n+1} 1, 0)$ in \mathbf{Y}_n , for $i = 0, 1, \dots, n$; see Figure 8.

It is easy to check the following facts, for $n \in \mathbb{N}$:

- every proper subalgebra of \mathbf{X}_n belongs to $\mathbb{ISP}^+(\mathbf{Y}) \subseteq \mathbb{ISP}^+(\mathbf{W})$;
- every homomorphism from \mathbf{X}_n into \mathbf{W} is constant;
- we can define a surjective homomorphism $\varphi_n : \mathbf{X}_{n+1} \rightarrow \mathbf{X}_n$ by $\varphi_n(i, y) := (i, y)$, for $0 \leq i \leq n$, and $\varphi_n(n+1, y) := (0, 0)$.

It follows that the conditions of the SILT 3.9 are satisfied. Hence \mathbf{W} is non-standard and not finitely uH-based. \square

Again, this proof works with trivial modifications if \mathbf{U} is replaced by any finite set of unary algebras, each with at most k elements. One can also extend the proof to show that $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{W})$ is not first-order definable (using the SUPT 5.3).

Corollary 8.13. *Let \mathbf{U} be a finite unary algebra of finite type with at least two operations. Then there is a chain of finite unary algebras $\mathbf{U} \leq \mathbf{U}_0 \leq \mathbf{U}_1 \leq \mathbf{U}_2 \leq \dots$ such that \mathbf{U}_i is finitely uH-based and standard, for i even, and \mathbf{U}_i is not finitely uH-based and non-standard, for i odd.*

9. CYCLIC SEMIGROUPS

The problems we have addressed in this paper appear to become much more difficult in the presence of non-unary operations. Nevertheless, in this section we give a complete answer to each of them for the uH-class generated by a finite cyclic semigroup, by establishing the following theorem.

Theorem 9.1. *For every finite cyclic semigroup \mathbf{S} , the following are equivalent:*

- (i) \mathbf{S} is standard;
- (ii) the topological prevariety generated by \mathbf{S} is first-order definable;
- (iii) \mathbf{S} is finitely uH-based;
- (iv) the variety generated by \mathbf{S} is residually small;
- (v) \mathbf{S} has index at most 2.

Recall that, if they exist, the **index** and **period** of a semigroup \mathbf{S} are the smallest positive numbers i and p such that \mathbf{S} satisfies $x^i \approx x^{i+p}$. Every finite semigroup has an index and period.

The equivalence of (iv) and (v) in Theorem 9.1 follows both from Golubov and Sapir [22] and from McKenzie [28]. Here, we will establish the remaining equivalences. We begin by proving two general theorems that give conditions under which a finite structure is standard and finitely uH-based. The first theorem is a generalization of one for unary algebras by Davey and Talukder [19]. Recall that the atomic class generated by a structure \mathbf{M} is $\text{At}(\mathbf{M}) = \mathbb{H}\text{SP}(\mathbf{M})$.

Theorem 9.2. *Let $\mathbf{M} = \langle M; G^{\mathbf{M}}, R^{\mathbf{M}} \rangle$ be a finite structure, with R finite. Assume that $\text{At}(\mathbf{M})$ has FDSC and that \mathbf{M} is injective in $\text{At}(\mathbf{M})$. Then \mathbf{M} is standard.*

Proof. Let \mathbf{X} be a Boolean topological model of $\text{Th}_{\text{uH}}(\mathbf{M})$. Then $\mathbf{X} \in \mathbb{I}\text{SP}^+(\mathbf{M})$. We shall use the Separation Theorem 1.3 to prove that $\mathbf{X} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$. Let $(a_1, \dots, a_m) \in X^m \setminus r^{\mathbf{X}}$, for some m -ary relation $r \in R \cup \{=\}$, and let \mathbf{Y} denote the substructure of \mathbf{X} generated by a_1, \dots, a_m . Since \mathbf{X} is locally finite, the structure \mathbf{Y} is finite. So $\mathbf{Y} \in [\mathbb{I}\text{SP}^+(\mathbf{M})]_{\text{fin}} \subseteq \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$.

For each choice of an n -ary $s \in R \cup \{=\}$ and $(b_1, \dots, b_n) \in Y^n \setminus s^{\mathbf{X}}$, we can find a clopen equivalence θ on \mathbf{X} for which $s^{\mathbf{X}} \cap (b_1/\theta \times \dots \times b_n/\theta) = \emptyset$, since $s^{\mathbf{X}}$ is closed in \mathbf{X}^n . As both \mathbf{Y} and R are finite, the intersection Θ of all of these clopen equivalences is a clopen equivalence. By Lemma 1.5, the congruence $\text{Syn}(\Theta)$ is also clopen. So $\mathbf{X}/\text{Syn}(\Theta)$ is finite, and the natural homomorphism $\eta : \mathbf{X} \rightarrow \mathbf{X}/\text{Syn}(\Theta)$ is continuous.

By construction, the restriction $\eta|_{\mathbf{Y}} : \mathbf{Y} \rightarrow \eta(\mathbf{X})$ is an embedding. Now, since we have $(\eta(a_1), \dots, \eta(a_m)) \notin r^{\eta(\mathbf{X})}$ and $\eta(\mathbf{Y}) \cong \mathbf{Y} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$, there exists a

morphism $\varphi : \eta(\mathbf{Y}) \rightarrow \mathbf{M}$ with $(\varphi \circ \eta(a_1), \dots, \varphi \circ \eta(a_m)) \notin r^{\mathbf{M}}$. As \mathbf{M} is injective in $\text{At}(\mathbf{M}) = \mathbb{H}\text{ISP}(\mathbf{M})$, this morphism extends to a morphism $\bar{\varphi} : \eta(\mathbf{X}) \rightarrow \mathbf{M}$. Now the morphism $\bar{\varphi} \circ \eta : \underline{\mathbf{X}} \rightarrow \mathbf{M}$ separates (a_1, \dots, a_m) and $r^{\mathbf{X}}$. Thus $\underline{\mathbf{X}} \in \mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{M})$. \square

In practice, you only need to check that \mathbf{M} is injective in $[\text{At}(\mathbf{M})]_{\text{fin}}$, and it then follows that \mathbf{M} is injective in $\text{At}(\mathbf{M})$; see [8, 2.2.9]. We next find that injectivity is also connected with finite axiomatizability.

Theorem 9.3. *Let $\mathbf{M} = \langle M; G^{\mathbf{M}}, R^{\mathbf{M}} \rangle$ be a finite structure, with R finite, and assume that \mathbf{M} is injective in $\text{At}(\mathbf{M})$. If $\text{At}(\mathbf{M})$ is finitely axiomatizable, then $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$ is finitely axiomatizable.*

Proof. Let k be a finite bound on the arity of the relation symbols in $R \cup \{=\}$. Assume that $\mathcal{A} := \text{At}(\mathbf{M})$ is finitely axiomatizable. We will first show that $\mathcal{M} := \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{M})$ can be defined within \mathcal{A} by the k -variable uH-theory of \mathbf{M} .

Let \mathbf{X} be a structure in \mathcal{A} that satisfies the k -variable uH-theory of \mathbf{M} . Then every substructure of \mathbf{X} that is generated by at most k elements belongs to \mathcal{M} . Now let $r \in R \cup \{=\}$ and $(a_1, \dots, a_m) \in X^m \setminus r^{\mathbf{X}}$. As $\text{Sg}_{\mathbf{X}}(\{a_1, \dots, a_m\}) \in \mathcal{M}$, there is a homomorphism $\varphi : \text{Sg}_{\mathbf{X}}(\{a_1, \dots, a_m\}) \rightarrow \mathbf{M}$ such that $(\varphi(a_1), \dots, \varphi(a_m)) \notin r^{\mathbf{M}}$. By injectivity, the homomorphism φ extends to a homomorphism from \mathbf{X} to \mathbf{M} that separates (a_1, \dots, a_m) and $r^{\mathbf{X}}$. Hence $\mathbf{X} \in \mathcal{M}$.

Now, as a basis for the uH-theory of \mathbf{M} , we may take a finite atomic basis Σ for \mathcal{A} and adjoin to it the k -variable uH-theory of \mathbf{M} . Let $\mathbf{M}^b = \langle M; G^{\mathbf{M}} \rangle$ denote the algebraic reduct of \mathbf{M} . Because the k -generated free algebra in $\text{Var}(\mathbf{M}^b)$ is finite, there are only finitely many ‘different’ k -variable terms in \mathcal{A} . So the k -variable uH-theory of \mathbf{M} is equivalent, in the presence of Σ , to a finite set of uH-formulae. \square

Lemma 9.4. *Let \mathbf{S} be a finite cyclic semigroup of index at most 2. Then \mathbf{S} is injective in the variety $\text{Var}(\mathbf{S})$ generated by \mathbf{S} .*

Proof. Every finite cyclic group is injective in the variety it generates; see [8, 4.4.2]. If \mathbf{S} has index 1, then it is a cyclic group, and so the result holds. Now assume that \mathbf{S} has index 2. Let p be the period of \mathbf{S} , and let s be the generator of \mathbf{S} . Then the set $\{s^2, \dots, s^{p+1}\}$ of all products in \mathbf{S} forms a cyclic subgroup \mathbf{G}_S of \mathbf{S} . Consider any $\mathbf{A} \in \text{Var}(\mathbf{S})$. Since \mathbf{S} satisfies $(xy)^{p+1} \approx xy$, the set $G_A := \{a \in A \mid a^{p+1} = a\}$ consists of all products in \mathbf{A} and therefore forms a subsemigroup \mathbf{G}_A of \mathbf{A} . In fact, we have $\mathbf{G}_A \in \text{Var}(\mathbf{G}_S)$, and so \mathbf{G}_A is a subgroup of \mathbf{A} .

Let $\mathbf{A}, \mathbf{B} \in \text{Var}(\mathbf{S})$ with $\mathbf{A} \leq \mathbf{B}$, and let $\varphi : \mathbf{A} \rightarrow \mathbf{S}$ be a homomorphism. The map φ restricts to a group homomorphism from \mathbf{G}_A to \mathbf{G}_S , and \mathbf{G}_A is a subgroup of \mathbf{G}_B . So we may use the injectivity of the cyclic group \mathbf{G}_S in $\text{Var}(\mathbf{G}_S)$ to find a homomorphism $\bar{\varphi} : \mathbf{G}_B \rightarrow \mathbf{G}_S$ extending $\varphi|_{\mathbf{G}_A}$. Let e denote the identity of $\mathbf{G}_A \leq \mathbf{G}_B$. We want to define $\psi : \mathbf{B} \rightarrow \mathbf{S}$ by $\psi(b) := \varphi(b)$, if $b \in \mathbf{G}_A$, and $\psi(b) := \bar{\varphi}(be)$, otherwise. For every $a \in \mathbf{G}_A$, we have $\psi(a) = \bar{\varphi}(ae) = \bar{\varphi}(a) = \varphi(a)$. So the map ψ is an extension of φ .

To verify that ψ is a homomorphism, let $b, c \in \mathbf{B}$. Then $bc, be, ce \in \mathbf{G}_B$ and so

$$\psi(bc) = \bar{\varphi}(bce) = \bar{\varphi}(bece) = \bar{\varphi}(be)\bar{\varphi}(ce).$$

First assume that $b, c \notin \mathbf{G}_A$. Then $\psi(bc) = \bar{\varphi}(be)\bar{\varphi}(ce) = \psi(b)\psi(c)$. Now we can assume that $b \in \mathbf{G}_A$ and $c \notin \mathbf{G}_A$. We have $be \in \mathbf{G}_A$ and so

$$\psi(bc) = \bar{\varphi}(be)\bar{\varphi}(ce) = \varphi(be)\psi(c) = \varphi(b)\varphi(e)\psi(c) = \psi(b)\psi(c),$$

as $\varphi(e)$ is the identity of \mathbf{G}_S and $\psi(c) = \bar{\varphi}(ce) \in \mathbf{G}_S$. \square

We mentioned in Section 1 that the variety of semigroups has FDSC. It is well known that the variety generated a cyclic semigroup has a finite basis for its equational theory. Indeed, this variety is commutative and so finitely axiomatizable, by a result of Perkins [31]. Theorems 9.2 and 9.3, together with Lemma 9.4, now show that (v) implies (i) and (iii) in Theorem 9.1. Since (i) implies (ii) trivially, it remains to show that $\neg(v)$ implies $\neg(ii)$ and $\neg(iii)$.

Fix a cyclic semigroup \mathbf{S} with index $i \geq 3$ and period $p \geq 1$. Let s be the generator of \mathbf{S} . The set $\{s^i, s^{i+1}, \dots, s^{i+p-1}\}$ forms a p -element cyclic subgroup of \mathbf{S} , which we denote by \mathbf{G}_S .

Sapir [34] showed that the cyclic semigroup with index 3 and period 1 is not finitely q-based. We are going to use a construction reminiscent of Sapir's to build two sequences of finite semigroups. We will be able to use these sequences to show that \mathbf{S} is not finitely uH-based and, via the SUPT 5.3, that $\mathbb{I}\mathbb{S}_c\mathbb{P}^+(\mathbf{S})$ is not first-order definable. This will complete the proof of Theorem 9.1.

Our first sequence will consist of finite semigroups that are not in $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{S})$ but that have large subsemigroups that are in $\mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{S})$. Let $n \in \mathbb{N}$ and let \mathbf{F}_n denote the free commutative semigroup on the generating set

$$A_n := \{a_{k,j} \mid 0 \leq k < n \text{ and } 1 \leq j < i\} \cup \{b_{k,j} \mid 0 \leq k < n \text{ and } 1 \leq j < i\}.$$

For each word $w \in F_n$, we let $|w|$ denote the length of w and let $c(w)$ denote the set of letters appearing in w (the **content** of w).

Now assume that $n \geq 2$. Define the equivalence relation θ_n on F_n by $(u, v) \in \theta_n$ if one of the following conditions holds:

- (1) $|u|, |v| \geq i$ and $|u| \equiv |v| \pmod{p}$;
- (2) there is some $0 \leq k < n$ such that u and v belong to the 3-element set

$$c_k := \{ a_{k,1} \cdots a_{k,i-1}, \quad b_{k,1} \cdots b_{k,i-1}, \quad a_{k \ominus 1,1} b_{k \ominus 1,2} \cdots b_{k \ominus 1,i-1} \},$$

where \ominus represents subtraction modulo n ;

- (3) $|u| = |v| < i$ and $c(u) = c(v)$.

It is straightforward to check that θ_n is a congruence on \mathbf{F}_n , and so we can define the semigroup $\mathbf{X}_n := \mathbf{F}_n / \theta_n$. The three parts of the definition of θ_n determine three different types of elements of X_n .

- (1) The set of words $g_\ell := \{w \in F_n \mid |w| \geq i \text{ and } |w| \equiv \ell \pmod{p}\}$ is a θ_n -class, for each $0 \leq \ell < p$. These p classes form a cyclic subgroup \mathbf{G}_n of \mathbf{X}_n .
- (2) The set c_k is a θ_n -class, for each $0 \leq k < n$.
- (3) The set of words in F_n of length ℓ and content B is a θ_n -class, for each $0 < \ell < i$ and $B \subseteq A_n$ with $0 < |B| \leq \ell$; unless $B = c(w)$, for some $w \in c_k$.

Notice, in particular, that \mathbf{X}_n is a finite commutative semigroup. In general, we will use words in F_n as names for the elements of X_n that they represent.

Lemma 9.5. *For each $n \geq 2$, we have $\mathbf{X}_n \notin \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{S})$.*

Proof. Choose any homomorphism $\varphi : \mathbf{X}_n \rightarrow \mathbf{S}$. We will show that $\varphi(c_0) = \varphi(c_1)$, and it will then follow that $\mathbf{X}_n \notin \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{S})$.

First assume that $\varphi(c_k) \notin G_S$, for some $k < n$. Using the first two representatives of c_k , we have

$$\varphi(a_{k,1})\varphi(a_{k,2}) \cdots \varphi(a_{k,i-1}) = \varphi(b_{k,1})\varphi(b_{k,2}) \cdots \varphi(b_{k,i-1}) = \varphi(c_k) \notin G_S.$$

Thus each $\varphi(a_{k,j})$ and $\varphi(b_{k,j})$ must be the generator s of \mathbf{S} , whence $\varphi(c_k) = s^{i-1}$. Define $k \oplus 1 := k \oplus_n 1$. Using the third representative of $c_{k \oplus 1}$, we obtain

$$\varphi(c_{k \oplus 1}) = \varphi(a_{k,1})\varphi(b_{k,2}) \cdots \varphi(b_{k,i-1}) = s^{i-1} = \varphi(c_k).$$

Since addition is modulo n , we can conclude by induction that $\varphi(c_0) = \varphi(c_1)$.

Now assume that $\varphi(c_k) \in G_S$, for all $k < n$. Since \mathbf{G}_S satisfies $x^{p+1} \approx x$, we can compute $\varphi(c_k) = \varphi(c_k)^{p+1} = \varphi(c_k^{p+1}) = \varphi(a^{(i-1)(p+1)})$, for any $a \in A_n$ and all $k < n$. Thus $\varphi(c_0) = \varphi(c_1)$. \square

We next show that \mathbf{X}_n has a large subsemigroup from $\mathbb{ISP}^+(\mathbf{S})$. Define Z_n to be the set of all elements of \mathbf{X}_n that have a representative from F_{n-1} . Then \mathbf{Z}_n is the subsemigroup of \mathbf{X}_n generated by the representatives in A_{n-1} . The set Z_n contains all the elements g_0, \dots, g_{p-1} of the cyclic group \mathbf{G}_n and all the elements c_0, c_1, \dots, c_{n-1} . Notice that the above induction argument would not work for \mathbf{Z}_n , due to the missing generators.

Lemma 9.6. *For each $n \geq 2$, we have $\mathbf{Z}_n \in \mathbb{ISP}^+(\mathbf{S})$.*

Proof. We want to show that the elements of \mathbf{Z}_n are separated by homomorphisms into \mathbf{S} . We begin by defining a collection of homomorphisms from \mathbf{Z}_n to \mathbf{S} .

Let B be a non-empty subset of A_{n-1} , and define \overline{B} to be the set of all elements of Z_n that can be represented by a word of length at most $i-1$ that only uses letters from B . We say that \overline{B} is **closed** if, for all $u, v \in Z_n$ such that $uv \in \overline{B}$, we have $u, v \in \overline{B}$. Notice that the closure condition can only fail if $uv = c_k$, for some $k < n$. It follows that \overline{B} is closed if and only if

- for all $k < n$ with $c_k \in \overline{B}$ and all $w \in c_k$ with $c(w) \subseteq A_{n-1}$, we have $c(w) \subseteq B$.

In particular: the set $\overline{A_{n-1}}$ is closed; the set \overline{B} is closed if, for all $k < n$, the set B does not include the content of any representative of c_k .

As \mathbf{S} satisfies $x^i \approx x^{i+p}$, we can define the map $\varphi_B : Z_n \rightarrow S$ by

$$\varphi_B(w) := \begin{cases} s^{|w|} & \text{if } w \in \overline{B}, \\ s^{ip+|w|} & \text{otherwise,} \end{cases}$$

where s is the generator of \mathbf{S} and $|w|$ is the length of any word representing w . We will show that φ_B is a homomorphism from \mathbf{Z}_n to \mathbf{S} , under the assumption that \overline{B} is closed. Let $u, v \in Z_n$. We consider two cases.

Case 1: $uv \in \overline{B}$. As \overline{B} is closed, we have $u, v \in \overline{B}$. Therefore $\varphi_B(uv) = s^{|uv|} = s^{|u|+|v|} = \varphi_B(u)\varphi_B(v)$.

Case 2: $uv \notin \overline{B}$. If $u, v \notin \overline{B}$, then we have $\varphi_B(uv) = s^{ip+|uv|} = s^{ip+|u|+ip+|v|} = \varphi_B(u)\varphi_B(v)$. If $u \notin \overline{B}$ and $v \in \overline{B}$, then $\varphi_B(uv) = s^{ip+|u|+|v|} = \varphi_B(u)\varphi_B(v)$. So we can now assume that $u, v \in \overline{B}$. Since $uv \notin \overline{B}$, we must have $|uv| \geq i$. Thus $\varphi_B(uv) = s^{ip+|uv|} = s^{|uv|} = s^{|u|+|v|} = \varphi_B(u)\varphi_B(v)$.

We have shown that $\varphi_B : \mathbf{Z}_n \rightarrow \mathbf{S}$ is a homomorphism, for each non-empty subset B of A_{n-1} such that \overline{B} is closed. Now let x and y be distinct elements of \mathbf{Z}_n . In order to separate x and y , we consider which of the three types of elements of \mathbf{X}_n they are: (1), (2) or (3).

If x is of type (1) and y is of type (2) or (3), then we choose $B := A_{n-1}$, which gives us $\varphi_B(x) \in G_S$ and $\varphi_B(y) \notin G_S$. If x is of type (2) and y is of type (3), then we choose $B := c(y)$, which again gives us $\varphi_B(x) \in G_S$ and $\varphi_B(y) \notin G_S$. Using symmetry, we are left with the cases in which x and y are of the same type.

Case 1: x and y are of type (1). Since $x \neq y$, the lengths of the words representing x and y are not congruent modulo p . Thus, for any closed \overline{B} , we have $\varphi_B(x) = s^{ip+|x|} \neq s^{ip+|y|} = \varphi_B(y)$.

Case 2: x and y are of type (2). Let us assume that $x = c_\ell$ and $y = c_m$, for some $0 \leq \ell < m < n$. Define $C := \{a_{m-1,1}, b_{m-1,2}, \dots, b_{m-1,i-1}\}$ and

$$D := \{a_{k,j} \mid m \leq k < n-1 \text{ and } 1 \leq j < i\} \\ \cup \{b_{k,j} \mid m \leq k < n-1 \text{ and } 1 \leq j < i\}.$$

It is straightforward to check that \overline{B} is closed, where $B := C \cup D \subseteq A_{n-1}$. We have $y = c_m \in \overline{B}$ and $x = c_\ell \notin \overline{B}$. So $\varphi_B(x) \in G_S$ and $\varphi_B(y) \notin G_S$.

Case 3: x and y are of type (3). All of the representatives of x have the same length, ℓ_x , and all of the representatives of y have the same length, ℓ_y . If $\ell_x < \ell_y$, then x and y are separated by φ_B , where $B := A_{n-1}$. So we can assume that $\ell_x = \ell_y$. Since $x \neq y$, it must be that $c(x) \neq c(y)$. We can further assume that $c(x) \not\subseteq c(y)$. Now define $B := c(y)$. This gives us $\varphi_B(x) \in G_S$ and $\varphi_B(y) \notin G_S$. \square

We can now prove that $\mathbb{ISP}^+(\mathbf{S})$ does not have a finite uH-basis.

Proposition 9.7. *Let \mathbf{S} be a finite cyclic semigroup of index greater than 2. Then \mathbf{S} is not finitely uH-based.*

Proof. Let $n \in \mathbb{N}$. We want to show that the n -variable uH-theory of \mathbf{S} does not serve to axiomatize $\mathbf{S} := \mathbb{ISP}^+(\mathbf{S})$. Define $m := 2^{ni}$, where i is the index of \mathbf{S} . We know that $\mathbf{X}_m \notin \mathbf{S}$, by Lemma 9.5. Let $Y \subseteq X_m$ with $|Y| \leq n$. Each element of \mathbf{X}_m can be represented by a word that uses at most $i-1$ letters of A_m . Consequently, at most $n(i-1)$ letters from A_m are needed to write representatives for all the elements of Y . Since $m = 2^{ni} > n(i-1)$, there is some $0 \leq k < m$ such that Y is contained in the subsemigroup of \mathbf{X}_m generated by the representatives from $A_m \setminus B_k$, where

$$B_k := \{a_{k,j} \mid 1 \leq j < i\} \cup \{b_{k,j} \mid 1 \leq j < i\}.$$

This subsemigroup of \mathbf{X}_m is isomorphic to \mathbf{Z}_m , which belongs to \mathbf{S} , by Lemma 9.6. Thus $\text{Sg}_{\mathbf{X}_m}(Y) \in \mathbf{S}$. We have shown that every subsemigroup of \mathbf{X}_m generated by at most n elements belongs to \mathbf{S} , but that \mathbf{X}_m does not belong to \mathbf{S} . It follows that \mathbf{S} is not axiomatized by the n -variable uH-theory of \mathbf{S} . \square

Remark 9.8. It is possible to set up an inverse system on $\{\mathbf{X}_{2^{ni}} \mid n \in \mathbb{N}\}$, and then to apply the SILT 3.9 to $\underline{\mathbf{X}} := \varprojlim \{\mathbf{X}_{2^{ni}} \mid n \in \mathbb{N}\}$ to show that \mathbf{S} is non-standard. Unfortunately, this inverse limit $\underline{\mathbf{X}}$ does not have a λ -space topology, for any $\lambda \in \mathbb{N}$. So we cannot apply the SUPT 5.3 to $\underline{\mathbf{X}}$ to show that $\mathbb{IS}_c\mathbb{P}^+(\mathbf{S})$ is not first-order definable. Instead, we will use the semigroups $\{\mathbf{Z}_n \mid n \geq 2\}$ to construct a different witness to the nonstandardness of \mathbf{S} , to which the SUPT will apply.

Proposition 9.9. *Let \mathbf{S} be a finite cyclic semigroup of index greater than 2. Then the topological prevariety $\mathbb{IS}_c\mathbb{P}^+(\mathbf{S})$ generated by \mathbf{S} is not first-order definable.*

Proof. For each $n \geq 2$, we can define an embedding $\psi_n : \mathbf{Z}_n \rightarrow \mathbf{Z}_{n+1}$ as follows: let $z \in Z_n$ and let w be a representative of z from F_{n-1} ; then $\psi_n(z)$ is the element of Z_{n+1} that also has w as a representative. We identify \mathbf{Z}_n with $\psi_n(\mathbf{Z}_n) \leq \mathbf{Z}_{n+1}$, so that we have a chain $\mathbf{Z}_2 \leq \mathbf{Z}_3 \leq \mathbf{Z}_4 \leq \dots$. We can now form the direct limit $\mathbf{Z} := \bigcup \{\mathbf{Z}_n \mid n \geq 2\}$. Since $\mathbf{Z}_n \in \mathbf{S} := \mathbb{ISP}^+(\mathbf{S})$, for each $n \geq 2$, we have $\mathbf{Z} \in \mathbf{S}$.

The direct limit \mathbf{Z} has a p -element cyclic subgroup \mathbf{G}_Z , corresponding to the group $\mathbf{G}_n \leq \mathbf{Z}_n$, for each $n \geq 2$. The representatives of \mathbf{G}_Z are all words of length at least i . We can define the homomorphism $f : \mathbf{Z} \rightarrow \mathbf{G}_Z$, via representative words, by $f(w) := a^{ip+|w|}$, where a is any letter from $\bigcup\{A_n \mid n \geq 2\}$. The map f is a p -compactification of Z onto G_Z , determining the p -space topology \mathcal{T}^f on Z . It is easy to check that \mathcal{T}^f is compatible with the semigroup operation \cdot of \mathbf{Z} . (It suffices to show that $\cdot^{-1}(z)$ is clopen, for all $z \in Z \setminus G_Z$, and that $\cdot^{-1}(f^{-1}(z))$ is open, for all $z \in G_Z$.) By enriching \mathbf{Z} with \mathcal{T}^f , we obtain a topological semigroup $\mathcal{Z} \in \mathcal{S}_{\text{Bt}}$.

We now want to show that \mathcal{Z} witnesses the non-standardness of \mathbf{S} . So we must prove that $\mathcal{Z} \notin \mathcal{S}_{\text{Bc}}$. Let e be the identity element of the group \mathbf{G}_Z . In \mathbf{Z} , we have $c_0 = \{a_{0,1} \cdots a_{0,i-1}, b_{0,1} \cdots b_{0,i-1}\}$ and

$$c_k = \{ a_{k,1} \cdots a_{k,i-1}, b_{k,1} \cdots b_{k,i-1}, a_{k-1,1} b_{k-1,2} \cdots b_{k-1,i-1} \},$$

for all $k \in \mathbb{N}$. The set G_Z is an ideal of \mathbf{Z} : $x \in G_Z$ and $y \in Z$ implies $xy \in G_Z$. So $ec_0 \in G_Z$. Since $c_0 \notin G_Z$, this gives us $c_0 \neq ec_0$. Now let $\varphi : \mathcal{Z} \rightarrow \mathbf{S}$ be any continuous homomorphism. We will show that $\varphi(c_0) = \varphi(ec_0)$. By the Separation Theorem 1.3, it will then follow that $\mathcal{Z} \notin \mathcal{S}_{\text{Bc}}$.

For all $k \geq 0$, we have $f(c_k) = f(c_0) = ef(c_0) = f(ec_0) = ec_0$, and therefore $\{c_0, c_1, c_2, \dots\} \subseteq f^{-1}(ec_0)$. As φ is continuous, this implies that $\varphi(c_k) = \varphi(ec_0) \in \varphi(G_Z) \subseteq G_S$, for some $k \geq 0$. If $\varphi(c_0) \notin G_S$, then the argument from the proof of Lemma 9.5 gives us $\varphi(c_k) \notin G_S$, which is a contradiction. We have established that $\varphi(c_0) \in G_S$. Thus $\varphi(c_0) = \varphi(e)\varphi(c_0) = \varphi(ec_0)$, whence $\mathcal{Z} \in \mathcal{S}_{\text{Bt}} \setminus \mathcal{S}_{\text{Bc}}$. Note that \mathbf{Z} is locally finite, since it belongs to $\mathcal{S} = \mathbb{I}\mathbb{S}\mathbb{P}^+(\mathbf{S})$.

Now let $\langle \mathbf{Y}; g \rangle$ be a model of the universal theory of $\langle \mathbf{Z}; f \rangle$. In order to be able to apply the SUPT 5.3, it remains to show that the semigroup operation \cdot of \mathbf{Y} is continuous with respect to the p -compactification topology \mathcal{T}^g . We will use the following universal properties of $\langle \mathbf{Y}; g \rangle$, inherited from $\langle \mathbf{Z}; f \rangle$:

- (1) the compactification $g : Y \rightarrow Y$ is an endomorphism of \mathbf{Y} ;
- (2) the set $g(Y)$ is an ideal of \mathbf{Y} ;
- (3) there is a finite bound $m \in \mathbb{N}$ such that, for each $y \in Y \setminus g(Y)$, there are at most m pairs $(u, v) \in Y^2$ with $uv = y$.

Now, to prove that \cdot is continuous, let U be an open set in \mathcal{T}^g and let $(x, y) \in \cdot^{-1}(U)$. We need to find open sets V_x and V_y in \mathcal{T}^g such that $(x, y) \in V_x \times V_y \subseteq \cdot^{-1}(U)$.

Case 1: $xy \notin g(Y)$. Since $g(Y)$ is an ideal of \mathbf{Y} , by (2), we must have $x, y \notin g(Y)$. Thus $\{x\}$ and $\{y\}$ are open in \mathcal{T}^g , with $\{x\} \cdot \{y\} \subseteq U$.

Case 2: $xy \in g(Y)$. As U is an open set containing xy , the set $W := g^{-1}(xy) \setminus U$ is finite. By (3), there are only finitely many pairs $(x', y') \in g^{-1}(g(x)) \times g^{-1}(g(y))$ with $x'y' \in W$. We now construct cofinite sets $V_x \subseteq g^{-1}(g(x))$ and $V_y \subseteq g^{-1}(g(y))$ by removing one member of each such pair (x', y') , being careful not to remove either x or y . The sets V_x and V_y are open in \mathcal{T}^g , with $(x, y) \in V_x \times V_y$. To see that $V_x \cdot V_y \subseteq U$, consider $x' \in V_x$ and $y' \in V_y$. Then $x'y' \notin W$. Since g is a homomorphism, by (1), we also have $g(x'y') = g(x')g(y') = g(x)g(y) = g(xy) = xy$. Thus $x'y' \in g^{-1}(xy) \setminus W = U$. \square

This completes the proof of Theorem 9.1. As an application of Theorem 3.14, we note that \mathcal{Z} could be realized as the inverse limit of a family of finite structures satisfying the conditions of the TILT 3.10. On the other hand, we could not see a way to obtain \mathcal{Z} as an application of the SILT 3.9.

10. OPEN PROBLEMS

Our Corollary 2.18 partitions finite structures into three types, according to the axiomatizability properties of the topological prevariety that they generate. In this paper, we have attempted to give a selection of interesting and non-trivial examples of each of these types. What we have not done is to give a useful characterization of the structures of any one of these types. Underlying this entire investigation is a general problem that appears to be very difficult.

Problem 1. *Is there an algorithm to decide, given a finite structure, whether the topological prevariety that it generates is standard, first-order definable but not universally definable, or not even first-order definable?*

This is clearly a problem that first needs to be restricted to special classes of finite structures. For example, we have answered it affirmatively for the class of finite cyclic semigroups.

A finite lattice is standard if the quasivariety that it generates is a variety [9]. In Section 4, we showed that the topological prevariety generated by a finite lattice is first-order definable, and then gave one example of a finite lattice that is nevertheless non-standard.

Problem 2. *Which finite lattices generate a standard topological prevariety?*

The examples we have seen suggest a strong correlation between the standardness of a finitely generated topological prevariety and the finite axiomatizability of the corresponding uH-class.

Problem 3. *If a finite structure generates a standard topological prevariety, must it be finitely uH-based?*

We have obtained an affirmative answer to the previous question for 3-element unary algebras and for cyclic semigroups. The lattice \mathbf{M}_{3-3} , which is not finitely uH-based, we showed to be non-standard as well. Possibly the most famous conjecture in variety theory, usually attributed to Park and Jónsson, is: ‘Every finite algebra \mathbf{M} of finite type such that $\text{Var}(\mathbf{M})$ contains only finitely many subdirectly irreducible algebras has a finite basis for its equational theory.’ If this is true, then every finite algebra \mathbf{M} such that $\text{Var}(\mathbf{M}) = \mathbb{ISP}(\mathbf{M})$ has a finite basis for its equational theory (and hence for its quasi-equational theory). So, if the conjecture is true, then the algebraic applications of Theorem 2.13 will all be to standard algebras that are finitely uH-based.

If we consider uH-classes that are not finitely generated, then we can answer Problem 3 in the negative.

Proposition 10.1. *There are uH-classes that are standard but not finitely based.*

Proof. Let \mathbf{M} be a finite algebra such that the variety $\mathcal{V} := \text{Var}(\mathbf{M})$ is not finitely based but has FDSC; for example, the 6-element semigroup proved to be not finitely based by Perkins [31]. Since \mathcal{V} is closed under quotients, it follows from Theorem 2.13 that \mathcal{V} is standard. \square

The converse of Problem 3 is not true. Section 6 gives a number of examples of relational structures that are finitely uH-based but non-standard, including the 2-element chain $\mathbf{2} = \langle \{0, 1\}; \leq \rangle$. Counterexample 3.7 of [10] gives a finite partial algebra that is finitely uH-based but non-standard. An example of a finite algebra that is finitely uH-based but non-standard has only recently been discovered [26].

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