Houston Journal of Mathematics

Volume 29, No. 4, 2003

STANDARD TOPOLOGICAL QUASI-VARIETIES

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Communicated by Klaus Kaiser

Abstract. This study addresses a problem which lies at the confluence of algebra, topology and mathematical logic. It is motivated by the theory of natural dualities, which provides a tight connection between a quasi-variety generated by a finite algebra $\underline{\mathbf{M}}$ and the topological quasi-variety generated by a related topological structure $\underset{\sim}{\mathbf{M}}$. We introduce the notion of a *standard topological quasi-variety* and initiate a program of study to determine which topological quasi-varieties are standard and which are not. We say that a topological quasi-variety is standard if, in an appropriate sense, there is a nice axiomatic description of its members. Knowing in advance that it is standard allows us to recognize its members by looking only at their finite substructures.

Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ be a finite structure with operations G, partial operations H, relations R and discrete topology $\mathcal{T}$. The *topological quasi-variety* generated by $\underset{\sim}{\mathbf{M}}$ is the category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{I}\mathbb{S}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ of isomorphic copies of topologically closed substructures of non-zero direct powers, with the product topology, of $\underset{\sim}{\mathbf{M}}$. Interest in topological quasi-varieties stems from the fact that they arise as the duals to algebraic quasi-varieties under natural dualities, and the name comes from their obvious structural similarity to algebraic quasi-varieties. (See Clark and Davey [4], Clark and Krauss [5].) A natural duality is a special kind of dual equivalence between the quasi-variety $\mathfrak{Q}(\underline{\mathbf{M}}) := \mathbb{I}\mathbb{S}\mathbb{P}\,\underline{\mathbf{M}}$ generated by a finite algebra $\underline{\mathbf{M}}$ and a topological quasi-variety $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{I}\mathbb{S}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ generated by a structure $\underset{\sim}{\mathbf{M}}$ having the same underlying set as $\underline{\mathbf{M}}$. The general theory of natural dualities provides methods to produce, from the algebra $\underline{\mathbf{M}}$, a structure $\underset{\sim}{\mathbf{M}}$ that will yield a natural duality on $\mathfrak{Q}(\underline{\mathbf{M}})$.

2000 *Mathematics Subject Classification.* 08C15, 08A60, 03B99, 06D50, 22A30.

Key words and phrases. Natural duality, topological quasi-variety, axiomatizability.

Often the structure $\underset{\sim}{\mathbf{M}}$ given by the general theory harbors a large and unwieldy collection of operations and relations, making it difficult to discern exactly which structures are in the category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ and what they look like. As a result, a major theme of duality theory is that of replacing $\underset{\sim}{\mathbf{M}}$ with a new dualising structure $\underset{\sim}{\mathbf{M}}'$ which is in some satisfactory sense simple enough to generate a comprehensible dual category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}')$. The literature reveals three different approaches that have been used to systematically identify and produce dualising structures that are acceptably simple.

(1) In [16] Davey and Werner gave a list of constructions that use existing operations and relations to generate new ones that are redundant for producing a duality. These constructions allowed them to eliminate structure from $\underset{\sim}{\mathbf{M}}$ until its operations and relations were reduced to an irredundant set. Davey, Haviar and Priestley [9] extended this list of constructions to a list that they could prove to be complete. (See [4], Chapter 9.)

(2) As a bottom up version of the previous approach, Davey and Priestley ([12], [13], [14]) gave inherent descriptions of those dualising structures from which no operation or relation could be eliminated without destroying the duality. Calling these *optimal dualities*, they gave efficient methods to produce them directly. (See [4], Chapter 8.)

(3) Clark and Davey [3] formulated a series of different restrictions on the type of the structure $\underset{\sim}{\mathbf{M}}$ that would make the structures in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ tractable; for example, having no partial operations or having only unary operations. They then characterized those algebras $\underline{\mathbf{M}}$ which admit a strong duality satisfying each of these restrictions. (See [4], Chapter 6.)

While each of these approaches has proven useful, there are choices of $\underset{\sim}{\mathbf{M}}$ satisfying any one of these conditions that still generate incomprehensible topological quasi-varieties. For example, Saramago [25] and Wegener [28] have both found means to press the limits of the optimal duality method. Taking $\underline{\mathbf{M}}$ to be the lattice $\mathbf{N}_5$, the NU Duality Theorem [4] gives us a dualising structure $\underset{\sim}{\mathbf{M}}$ with 5896 binary relations. Wegener showed that $\underset{\sim}{\mathbf{M}}$ can be reduced to an optimal dualising structure $\underset{\sim}{\mathbf{M}}'$ with only 4 binary relations. While this constitutes a phenomenal reduction, it is by no means clear that there is any practical way to tell which 4-relation structures $\mathbf{X}'$ are in the category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}')$.

Examples like this suggest that whatever restrictions we make on the structure $\underset{\sim}{\mathbf{M}}$, the ultimate goal remains the same. In order for the dual structures in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ to be considered "comprehensible", there should be some way to tell which structured spaces of the right type are in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ and to say what they look like. These structures are characterized by the existence of separating morphisms into $\underset{\sim}{\mathbf{M}}$.

Separation Theorem ([4], 1.4.4) *Let $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T} \rangle$ be a finite structure, let $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$, and let $\mathbf{X}$ be a compact topological structure of the same type as $\underset{\sim}{\mathbf{M}}$. Then $\mathbf{X} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ if and only if there is at least one morphism from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}$ and the following conditions hold:*

(i) *for each $x, y \in X$ where $x \neq y$, there is an $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $\alpha(x) \neq \alpha(y)$,*
(ii) *for each n-ary $h \in H$ and $(x_1, x_2, \dots, x_n) \in X^n \setminus \mathrm{dom}(h^{\mathbf{X}})$, there is an $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n)) \notin \mathrm{dom}(h^{\underset{\sim}{\mathbf{M}}})$,*
(iii) *for each n-ary $r \in R$ and $(x_1, x_2, \dots, x_n) \in X^n \setminus r^{\mathbf{X}}$, there is an $\alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ such that $(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n)) \notin r^{\underset{\sim}{\mathbf{M}}}$.*

While we will make frequent use of this basic fact, it rarely gives us a readily verifiable condition for membership in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$.

What is really needed is an *axiomatic* description of the members of $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. To find this description we are guided by the method used in the absence of topology. Let $\mathbf{M} = \langle M; G, H, R \rangle$. In the first-order language of $\mathbf{M}$, a *universal Horn sentence* is a universally quantified expression of one of the forms

$$\chi \quad \text{or} \quad \bigvee_{i \in I} \neg\psi_i \quad \text{or} \quad \bigwedge_{i \in I} \psi_i \Rightarrow \chi \tag{$*$}$$

where χ and each ψ_i are atomic formulas and I is a finite set. For example, consider the three-element unary algebra $\mathbf{N} = \langle \{0, 1, 2\}; g \rangle$, where g is the 3-cycle $(0\,1\,2)$. Then the universal Horn sentences

$$g^3(x) \approx x, \qquad g(x) \not\approx y \ \vee \ g(y) \not\approx x \qquad \text{and} \qquad g(x) \approx g(y) \implies x \approx y$$

all hold in $\mathbf{N}$. Notice that, in general, the first form of universal Horn sentence is the special case of the third where $I = \varnothing$. The second is also a special case of the third if there is a choice of χ which is known to be false; for example, take χ to be $g(x) \approx x$ for $\mathbf{N}$. (Note that disjunctions of negations of atomic formulas never hold in $\mathbb{ISP}\,\mathbf{M}$, since the empty power is included. The only difference between the quasi-variety $\mathbb{ISP}\,\mathbf{M}$ and the universal Horn class $\mathbb{ISP}^+\mathbf{M}$ is that $\mathbb{ISP}\,\mathbf{M} = \mathbb{ISP}^+\,\mathbf{M} \cup \{\mathbf{M}^\varnothing\}$. In particular, they are the same if it happens that $\mathbf{M}^\varnothing \in \mathbb{ISP}^+\,\mathbf{M}$.)

The universal Horn sentences that hold in $\mathbf{M}$ form the *universal Horn theory* of $\mathbf{M}$ which we denote by $\mathrm{Th}_{uH}(\mathbf{M})$. A classical theorem of Mal'cev [22] says that, if $\mathbf{M}$ is an algebra, then $\mathbb{ISP}^+\mathbf{M}$ is exactly the class of models of $\mathrm{Th}_{uH}(\mathbf{M})$;

in symbols,

$$\mathbb{ISP}^{+}\mathbf{M} = \operatorname{Mod}(\operatorname{Th}_{uH}(\mathbf{M})). \tag{$**$}$$

It turns out that the same is true of an arbitrary finite structure $\mathbf{M} = \langle M; G, H, R\rangle$ with partial operations and relations. (See Grätzer and Lakser [17], Andréka and Németi [1], McNulty [23].)

The advent of the theory of natural dualities raised the question as to whether there was a comparable syntactic description of the category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}$ generated by a finite discretely topologized structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\,\rangle$. This was answered affirmatively in Clark and Krauss [5] by formulating an infinitary version of univeral Horn logic which allows assertions involving both algebra and topology. More specifically, the authors defined a topological atomic formula to be either a standard (first-order) atomic formula or a special kind of convergence formula asserting that a set of terms, indexed over an arbitrary directed set, converges to a given term. They then defined a *topological universal Horn sentence* (which they called a 'topological quasi-atomic formula') to be a universally quantified expression of the form given by $(*)$ where χ and each ψ_i are topological atomic formulas and I is an arbitrary set. Armed with this notion they proved that, for every choice of a finite structure $\underset{\sim}{\mathbf{M}}$, the class $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}$ is characterized as the compact models of the topological universal Horn theory of $\underset{\sim}{\mathbf{M}}$; in symbols,

$$\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \operatorname{Mod}(\operatorname{Th}_{tuH}(\underset{\sim}{\mathbf{M}})).$$

While this characterization of topological quasi-varieties does not appear to have led to any useful description of the members of a particular topological quasi-variety, it is nevertheless conceptually helpful as it does lead to a new criterion for a topological quasi-variety to be considered "comprehensible".

Our new criterion requires another result which we quickly review. By a *Boolean structure* (of type $\langle G, H, R\rangle$) we mean a topological structure $\mathbf{X} = \langle X; G^{\mathbf{X}}, H^{\mathbf{X}}, R^{\mathbf{X}}, \mathcal{T}^{\mathbf{X}}\,\rangle$ such that

(i) $\langle X; \mathcal{T}^{\mathbf{X}}\,\rangle$ is a Boolean space,
(ii) if $h \in G \cup H$ is n–ary, then the domain $\operatorname{dom}(h^{\mathbf{X}})$ is a closed subset of X^n and $h^{\mathbf{X}} : \operatorname{dom}(h^{\mathbf{X}}) \to X$ is continuous, and
(iii) if $r \in R$ is n–ary, then $r^{\mathbf{X}}$ is a closed subset of X^n.

If Σ is a set of universal Horn sentences, we denote by $\operatorname{Mod}_{\mathcal{T}}(\Sigma)$ the class of all Boolean structures which satisfy each universal Horn sentence in Σ. (See [4], 1.4.)

Preservation Theorem ([4]: 1.4.3) *Let* $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\,\rangle$ *be a finite structure and let* $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}$. *Then every member of* $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ *is a Boolean* *model of the universal Horn theory of* $\underset{\sim}{\mathbf{M}}$*, in symbols,*

$$\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) \subseteq \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})).$$

In particular, for every choice of $\underset{\sim}{\mathbf{M}}$ each member of $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is a Boolean structure of type $\langle G, H, R\rangle$. It will therefore be convenient to view the category of all Boolean structures with continuous homomorphisms as the umbrella category in which we will do our work.

We can now define the central notion of this study. We will say that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is a *standard topological quasi-variety*, or that $\underset{\sim}{\mathbf{M}}$ is *standard*, if $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is exactly the class of all Boolean models of the standard universal Horn theory of $\underset{\sim}{\mathbf{M}}$; in symbols,

$$\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})).$$

Standardness tells us that membership in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is essentially determined by first-order algebraic properties alone. In fact standardness is an inherent property of $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$, independent of the particular choice of generator $\underset{\sim}{\mathbf{M}}$. For if $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_{c}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}_1 = \mathbb{IS}_{c}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}_2$, then the Preservation Theorem implies that $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_1) = \mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_2)$. Thus $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_1))$ if and only if $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_2))$. Note that a more accurate but considerably less user friendly name for these categories might have been "standard compact topological universal Horn classes". Following [5] we have opted instead for the name given in our title.

When a class of structures is defined by first-order properties, a useful description of those structures is often given by a set of axioms. We say that a subset $\Sigma \subseteq \mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$ *axiomatizes* $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ provided that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathrm{Mod}_{\mathcal{T}}(\Sigma)$. If $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is axiomatizable, then it is certainly standard, and the axioms provide a description of its members. Davey and Werner [16] gave a number of examples of topological quasi-varieties which they showed to be standard by exhibiting a small finite set Σ that axiomatized them. These and other examples of standard topological quasi-varieties will be reviewed in Section 2.

Our thesis here is that we can certainly claim to understand what is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ if we can write down a set Σ of universal Horn axioms for $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. Since an axiomatization exists if and only if $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard, the class $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ must be standard in order to be "comprehensible" in this sense. But standardness offers us even more. While all past proofs of standardness have been obtained by *producing* an axiom system, the process of establishing an axiomatization of $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is considerably simplified if we know *in advance* that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard. For suppose that we know $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard, and let $\Sigma \subseteq \mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$. In Corollary 1.4 we will see that Σ axiomatizes $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ provided only that each model of Σ is locally finite and each finite model of Σ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. Thus standardness tells us that we can axiomatize $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ *without any reference to topology.* Once we have an axiomatization Σ for $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$, we can decide if a Boolean structure $\mathbf{X}$ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ by simply checking that $\mathbf{X}$ is locally finite and that each finite substructure of $\mathbf{X}$ satisfies Σ, again with no reference to topology. These observations motivate the following question.

Standardness Problem *Which finite structures* $\underset{\sim}{\mathbf{M}}$ *generate a standard topological quasi-variety* $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}$?

Taking the first steps to solve this general problem, we shall exhibit many examples of topological quasi-varieties $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ which are standard and many which are not. The standardness problem for unary algebras is of particular interest since natural dualities with unary duals are among the most useful dualities. (See [4], Chapter 6.) As an illustration of our program, we will give a general proof that $\underset{\sim}{\mathbf{M}} = \langle M; f, \mathcal{T}\rangle$ is always standard if f is a single unary operation. Using this fact, we will show that a particular finite set Σ axiomatizes $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}$ by examining only the finite members of $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$.

1. General criteria for standardness

We continue to consider a finite structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$, with operations G, partial operations H, relations R and discrete topology $\mathcal{T}$, together with the topological quasi-variety $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}$ that it generates. If $\mathcal{M}$ is a class of structures, we denote by $\mathcal{M}_{\mathrm{fin}}$ the class of finite members of $\mathcal{M}$. If $\mathbf{X}$ is a Boolean structure and Σ is a set of universal Horn sentences, then $\mathbf{X} \models \Sigma$ means that $\mathbf{X}$ satisfies each sentence in Σ. We say that a Boolean structure $\mathbf{X} = \langle X; G, H, R, \mathcal{T}\rangle$ is *locally finite* if the partial algebra $\mathbf{X}' = \langle X; G, H\rangle$ is locally finite.

Since finite Boolean structures carry only the discrete topology, the result $(**)$ above tells us that every *finite* Boolean model of $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$ is in the topological quasi-variety generated by $\underset{\sim}{\mathbf{M}}$, that is,

$$\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{\mathrm{fin}} = \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}))_{\mathrm{fin}}.$$

In this sense $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ can always be thought of as being *finitely standard.* We will use this fact to give criteria for $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ to be (fully) standard.

Lemma 1.1 *If* $\mathbf{X}$ *is a Boolean structure of the same type as* $\underset{\sim}{\mathbf{M}}$*, then the following are equivalent:*

(i) $\mathbf{X} \models \mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$;
(ii) $\mathbf{X}$ *is locally finite and every finite substructure of* $\mathbf{X}$ *is in* $\mathfrak{Q}_{\mathfrak{T}}(\underset{\sim}{\mathbf{M}})$.

Proof. Assume (i). Then every finite substructure of $\mathbf{X}$ is also a model of $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$. Since $\mathfrak{Q}_{\mathfrak{T}}(\underset{\sim}{\mathbf{M}})$ is finitely standard, every finite substructure of $\mathbf{X}$ is in $\mathfrak{Q}_{\mathfrak{T}}(\underset{\sim}{\mathbf{M}})$. To see that $\mathbf{X}$ is locally finite, let $\mathbf{X}'$ and $\underset{\sim}{\mathbf{M}}'$ be obtained from $\mathbf{X}$ and $\underset{\sim}{\mathbf{M}}$, respectively, by deleting their relations and topologies. We will show that $\mathbf{X}'$ is locally finite. By [1], $\mathbf{X}' \in \mathbb{ISP}\,\underset{\sim}{\mathbf{M}}'$, so we must check that each power $(\underset{\sim}{\mathbf{M}}')^I$ of $\underset{\sim}{\mathbf{M}}'$ is locally finite. Consider a finite set $Z \subseteq M^I$ where $|Z| = n$. For $i, j \in I$, define $i \equiv j$ if $z(i) = z(j)$ for all $z \in Z$. The equivalence $\equiv$ partitions I into at most $|M|^n$ classes, where each member of Z is constant on each class. Thus Z is contained in a substructure of $(\underset{\sim}{\mathbf{M}}')^I$ that is isomorphic to $(\underset{\sim}{\mathbf{M}}')^{|M|^n}$, and is therefore finite. This proves (ii).

Now assume (ii) and consider φ as given in $(*)$. We first take

$$\varphi := \bigwedge_{i\in I} \psi_i(x_1, \dots, x_n) \to \chi(x_1, \dots, x_n)$$

in $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$. Pick $a_1, \dots, a_n \in X$ such that $\psi_i(a_1, \dots, a_n)$ is true in $\mathbf{X}$ for each $i \in I$. Since $\mathbf{X}$ is locally finite it has a finite substructure $\mathbf{Y}$ containing $\{a_1, \dots, a_n\}$. By (ii) we have $\mathbf{Y} \in \mathfrak{Q}_{\mathfrak{T}}(\underset{\sim}{\mathbf{M}})$. By the Preservation Theorem, $\mathbf{Y} \models \varphi$ so $\chi(a_1, \dots, a_n)$ is true in $\mathbf{Y}$ and therefore also in $\mathbf{X}$. Thus $\mathbf{X} \models \varphi$.

It remains to consider a disjunction of neg-atomic formulas

$$\bigvee_{i\in I} \neg\psi_i(x_1, \dots, x_n)$$

in $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$ where I is a finite set. Let $a_1, \dots, a_n \in X$ and assume $\psi_i(a_1, \dots, a_n)$ is true in $\mathbf{X}$ for each $i \in I$. Let $\mathbf{Y}$ be a finite substructure of $\mathbf{X}$ containing $a_1, \dots, a_n$. Then $\psi_i(a_1, \dots, a_n)$ is true in $\mathbf{Y}$ for each $i \in I$. By (ii), we have $\mathbf{Y} \in \mathfrak{Q}_{\mathfrak{T}}(\underset{\sim}{\mathbf{M}})$. By the Separation Theorem, there is a morphism $\alpha : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$. Since each ψ_i is an atomic formula we conclude that $\psi_i(\alpha(a_1), \dots, \alpha(a_n))$ is true in $\underset{\sim}{\mathbf{M}}$, a contradiction. ☐

This lemma provides us with a useful criterion for standardness that will be employed throughout this study.

Corollary 1.2 *Let* $\underset{\sim}{\mathbf{M}}$ *be a finite structure. Then* $\mathfrak{Q}_{\mathfrak{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$ *is standard if and only if each locally finite Boolean structure* $\mathbf{X}$ *whose finite substructures are all in* $\mathfrak{Q}_{\mathfrak{T}}(\underset{\sim}{\mathbf{M}})$ *is itself in* $\mathfrak{Q}_{\mathfrak{T}}(\underset{\sim}{\mathbf{M}})$.

Since we will be exhibiting examples of non-standard topological quasi-varieties, it will be helpful to explicitly state the contrapositive of Corollary 1.2.

Corollary 1.3 *Let $\underset{\sim}{\mathbf{M}}$ be a finite structure. Then $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ is non-standard if and only if there is a locally finite Boolean structure $\mathbf{X}$ such that each finite substructure of $\mathbf{X}$ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ but $\mathbf{X}$ itself is not in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$.*

If we can prove *in advance* that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is standard, then Corollary 1.2 allows us to establish that a set $\Sigma \subseteq \mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$ axiomatizes $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ without reference to topology.

Corollary 1.4 *Let $\underset{\sim}{\mathbf{M}}$ be a finite structure which generates a standard topological quasi-variety $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$. Then $\Sigma \subseteq \mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$ axiomatizes $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ provided every model of Σ is locally finite and each finite model of Σ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$.*

We will see that the following principle can be used to establish in advance that many choices of $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ are standard without (even implicitly) finding a set of axioms for $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. Our applications will be made to total algebras, where (ii) and (iii) are not necessary.

Lemma 1.5 *Let $\underset{\sim}{\mathbf{M}}$ be a finite structure. Then $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) := \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{M}}$ is standard provided that, for every $\mathbf{X} \in \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}))$,*

- (i) *for each $x, y \in X$ where $x \neq y$, there is a finite $\mathbf{Y} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ and an $\alpha : \mathbf{X} \to \mathbf{Y}$ such that $\alpha(x) \neq \alpha(y)$,*
- (ii) *for each n-ary $h \in H$ and $(x_1, x_2, \dots, x_n) \in X^n \setminus \mathrm{dom}(h^{\mathbf{X}})$, there is a finite $\mathbf{Y} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ and an $\alpha : \mathbf{X} \to \mathbf{Y}$ such that $(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n)) \notin \mathrm{dom}(h^{\mathbf{Y}})$,*
- (iii) *for each n-ary $r \in R$ and $(x_1, x_2, \dots, x_n) \in X^n \setminus r^{\mathbf{X}}$, there is a finite $\mathbf{Y} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ and an $\alpha : \mathbf{X} \to \mathbf{Y}$ such that $(\alpha(x_1), \alpha(x_2), \dots, \alpha(x_n)) \notin r^{\mathbf{Y}}$.*

Proof. We show that $\mathbf{X} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ by using each hypothesis to verify the corresponding part of the Separation Theorem. In each case, let $\alpha : \mathbf{X} \to \mathbf{Y}$ be the given separating morphism. Since $\mathbf{Y} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ there is, by the Separation Theorem, a separating morphism $\beta : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}$ and consequently $\beta \circ \alpha : \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ provides the required separation. ∎

2. Standard and non-standard examples

In order to provide a meaningful context for our study, we examine a number of familiar topological quasi-varieties to see which are standard and which are not.

We begin by giving a list of examples known to be standard. In each case, we give a reference to the original source and (if applicable) to an item in [4], where we refer the reader for details. These proofs of standardness are all established by listing a set Σ of axioms satisfied by the generator $\underset{\sim}{\mathbf{M}}$ and then showing that every Boolean model $\mathbf{X}$ of Σ has the required separating morphisms into $\underset{\sim}{\mathbf{M}}$.

Example 2.1 The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ of *Boolean spaces* $\mathbf{X} = \langle X; \mathcal{T} \rangle$ is obtained by taking $\Sigma = \varnothing$. If M is any finite set with more than one element, then $\underset{\sim}{\mathbf{M}} := \langle M; \mathcal{T} \rangle$ will serve as a generator for $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$, and $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is strongly dual to the variety generated by any primal algebra. (See Stone [26], [4]: 4.1.2; Hu [19], [4]: 4.1.1.)

Example 2.2 The category of *pointed Boolean spaces* $\mathbf{X} = \langle X; 0, \mathcal{T} \rangle$ is again obtained by taking $\Sigma = \varnothing$. For every prime p, this category is generated by $\underset{\sim}{\mathbf{GF}}(p) := \langle \{0, 1, \dots, p-1\}; 0, \mathcal{T} \rangle$ and is strongly dual to the quasi-variety generated by the field $\underline{\mathbf{GF}}(p) := \langle \{0, 1, \dots, p-1\}; +, \cdot, 0 \rangle$. (See [4]: 4.1.3.)

Example 2.3 The category of *Boolean rectangular bands* $\mathbf{X} = \langle X; *, \mathcal{T} \rangle$ is axiomatized by

$$\Sigma = \{x * (y * z) \approx (x * y) * z, \quad x * y \approx y * x \implies x \approx y\}.$$

It is generated by the six-element Boolean band $\underset{\sim}{\mathbf{2}} \times \underset{\sim}{\mathbf{3}}$ and is strongly dual to the quasi-variety generated by the ring with identity $\underline{\mathbf{Z}}_6$. (See [4]: 4.2.6. This result has been generalized by Davey and Knox [11].)

Example 2.4 The category of *Boolean abelian groups of exponent* m consists of all Boolean structures $\mathbf{X} = \langle X; +, 0, \mathcal{T} \rangle$ satisfying the axioms

$$\Sigma = \{x + (y + z) \approx (x + y) + z, \quad x + y \approx y + x, \quad x + 0 \approx x, \quad mx \approx 0\}.$$

This category is generated by the Boolean cyclic group of order m and is strongly dual to the variety of all abelian groups of exponent m. (See [16]; [4]: 4.4.2.)

Example 2.5 The category of *Boolean vector spaces over* F, where F is a finite field, is axiomatized by the usual vector space axioms. It is generated by $\underset{\sim}{\mathbf{F}}$, the one dimensional Boolean vector space over F, and is strongly dual to the variety of all vector spaces over F. (See [16]; [4]: 4.4.4.)

Example 2.6 The category of *Boolean meet semilattices with 1* consists of all Boolean structures $\mathbf{X} = \langle X; \cdot, 1, \mathcal{T} \rangle$ satisfying the axioms

$$\Sigma = \{x \cdot (y \cdot z) \approx (x \cdot y) \cdot z, \quad x^2 \approx x, \quad x \cdot y \approx y \cdot x, \quad x \cdot 1 \approx x\}.$$

This category is generated by the two-element Boolean semilattice with 1. It is strongly dual to the variety of all meet semilattices with 1. (See [18]; [4]: 4.4.7.)

Example 2.7 A large class of standard topological quasi-varieties is given in Davey and Werner [16]: the duals of varieties generated by *quasi-primal algebras*. Let $\underline{\mathbf{M}}$ be any finite algebra, and define H to be the inverse semigroup of all isomorphisms between subalgebras of $\underline{\mathbf{M}}$ together with the empty map. A *Boolean H-space* is defined to be a Boolean space $\mathbf{X} = \langle X; E, H, \mathcal{T} \rangle$ containing a set E of constants corresponding to one element subalgebras of $\underline{\mathbf{M}}$ and acted on by H in a way that satisfies a set Σ of quasi-equations listed in [5]. The authors show that the category of Boolean H-spaces is strongly dual to the variety generated by the quasi-primal algebra obtained from $\underline{\mathbf{M}}$ by adding the ternary discriminator operation. (See also [4]: 3.3.13.)

In contrast with these standard examples, most other familiar topological quasi-varieties $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ arise as the strong dual of a quasi-variety of lattice-ordered algebras. In these cases an order $\leqslant$ is often taken as part of the type of the structures in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. An ordered Boolean space $\mathbf{X} = \langle X; \leqslant, \mathcal{T} \rangle$ is said to be *totally order-disconnected* if, whenever $x, y \in X$ and $x \nleqslant y$, there is a clopen increasing subset of $\mathbf{X}$ containing x but not y. The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ of all compact totally order-disconnected spaces was found by Hilary Priestley [24] to be the strong dual of the variety of bounded distributive lattices, and the members of $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ are normally referred to as *Priestley spaces*. (See Davey and Priestley [15].)

Notice that the definition of "totally order-disconnected" is equivalent to the existence of separating morphisms into the chain $\underset{\sim}{\mathbf{2}} = \langle \{0, 1\}; \leqslant, \mathcal{T} \rangle$. Thus $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is exactly the topological quasi-variety $\mathbb{IS}_c\mathbb{P}^+ \, \underset{\sim}{\mathbf{2}}$. By the Preservation Theorem, each $\mathbf{X} = \langle X; \leqslant, \mathcal{T} \rangle$ in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is a Boolean ordered space, that is, a Boolean space with a topologically closed order. This leads to a natural question. Is every Boolean ordered space a Priestley space? If so, $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ would be a standard topological quasi-variety axiomatized by the set Σ of axioms for an ordering. In [27], Stralka produced an example which showed that this is not the case.

Example 2.8 *The category of Priestley spaces is not standard.* ([27]; [4]: 1.4.6.)

PROOF. Let $\langle C; \mathcal{T} \rangle$ be the Cantor space obtained by removing open middle thirds from the unit interval, that is, from $[0,1]$ we remove the open interval $(\frac{1}{3}, \frac{2}{3})$, then $(\frac{1}{9}, \frac{2}{9})$ and $(\frac{7}{9}, \frac{8}{9})$, and so forth. The remaining set C has a countable sequence of covering pairs $\frac{1}{3} < \frac{2}{3}$, $\frac{1}{9} < \frac{2}{9}$, $\frac{7}{9} < \frac{8}{9}, \dots$ under the usual order. We define the *Stralka order* $\leqslant$ on C by $x \leqslant y$ if either $x = y$ or y covers x in the usual order; for example, $\frac{1}{9} \leqslant \frac{2}{9}$ but $\frac{1}{9}$ and $\frac{1}{3}$ are not comparable. Since $\leqslant$ is closed in $C \times C$, the ordered space $\mathbf{X} = \langle C; \leqslant, \mathcal{T} \rangle$ is a Boolean structure. Every finite subspace of $\mathbf{X}$ is a Priestley space since its topology is discrete, and $\mathbf{X}$ is locally finite since it has no operations. The only clopen increasing subsets of $\mathbf{X}$ are intervals $[x, 1]$ where x is the right endpoint of a deleted third. Thus $\mathbf{X}$ is not a Priestley space itself since, for example, $\frac{1}{4} \nleqslant \frac{3}{4}$, but there is no clopen increasing set containing $\frac{1}{4}$ but not $\frac{3}{4}$. By Corollary 1.3 we see that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is not standard. □

Stralka's example tells us that, in the case of Priestley spaces, we can not replace total order-disconnectedness with universal Horn sentences. Building on Stralka's example, we will show that other familiar categories dual to lattice-ordered algebras are also non-standard by constructing appropriate Stralka-based Cantor-examples.

Example 2.9 *The category* $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{S}}$*, strongly dual to Stone algebras, is non-standard.*

PROOF. From Davey [7] ([4]: 4.3.7) we have $\underset{\sim}{\mathbf{S}} = \langle \{0, a, 1\}; d, \preccurlyeq, \mathcal{T} \rangle$ where $\preccurlyeq$ is the order with $1 \preccurlyeq a$ and 0 non-comparable to a and 1. For each $x \in S$, we define $d(x)$ to be the unique minimal element below x. The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ consists of all Boolean structures $\mathbf{X} = \langle X; d, \preccurlyeq, \mathcal{T} \rangle$ such that $\langle X; \preccurlyeq, \mathcal{T} \rangle$ is a Priestley space and $d(x)$ is the unique minimal element below x.

To see that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard, let $\mathbf{X} = \langle C; d, \preccurlyeq, \mathcal{T} \rangle$ where $\langle C; \mathcal{T} \rangle$ is the Cantor space, $\preccurlyeq$ is the Stralka order together with all pairs $(0, x)$ for $x \in C$, and $d : C \to \{0\}$ is the constant map. Clearly $\mathbf{X}$ is locally finite, d is continuous and every finite substructure of $\mathbf{X}$ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. But, as in Example 2.8, the space $\langle C; \preccurlyeq, \mathcal{T} \rangle$ is not a Priestley space so $\mathbf{X} \notin \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. Thus $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard by Corollary 1.3. □

Example 2.10 *The category* $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_c\mathbb{P}^+ \underset{\sim}{\mathbf{DS}}$*, strongly dual to double Stone algebras, is non-standard.*

PROOF. From Davey [7] ([4]: 4.3.14) we have $\underset{\sim}{\mathbf{DS}} = \langle \{0, a, b, 1\}; d, u, \preccurlyeq, \mathcal{T} \rangle$ where $\preccurlyeq$ is the order with $a \preccurlyeq b$ and no other comparabilities, where $d(x)$ is the unique minimal element below x and where $u(x)$ is the unique maximal element above x. The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ consists of all Boolean structures $\mathbf{X} = \langle X; d, u, \preccurlyeq, \mathcal{T}\rangle$ such that $\langle X; \preccurlyeq, \mathcal{T}\rangle$ is a Priestley space, $d(x)$ is the unique minimal element below x and $u(x)$ is the unique maximal element above x.

To show that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard, let $\mathbf{X} = \langle C; d, u, \preccurlyeq, \mathcal{T}\rangle$ where $\langle C; \mathcal{T}\rangle$ is the Cantor space, $\preccurlyeq$ is the Stralka order together with all pairs $(0, x)$ and $(x, 1)$ for $x \in X$. Let $d : C \to \{0\}$ and $u : C \to \{1\}$ be the constant maps. Clearly $\mathbf{X}$ is locally finite, d and u are continuous, and every finite substructure of $\mathbf{X}$ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. Since the space $\langle C; \preccurlyeq, \mathcal{T}\rangle$ is not a Priestley space, $\mathbf{X} \notin \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ and therefore $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard by Corollary 1.3. □

Example 2.11 *The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{DM}}$, strongly dual to De Morgan algebras, is non-standard.*

Proof. By Cornish and Fowler [6] ([4]: 4.3.16) we let $\underset{\sim}{\mathbf{DM}} = \langle \{0, a, b, 1\}; f, \preccurlyeq, \mathcal{T}\rangle$ where $\preccurlyeq$ is the non-linear lattice order with a at the top and b at the bottom, and f is the bijection fixing 0 and 1 and interchanging a and b. The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ consists of all Boolean structures $\mathbf{X} = \langle X; f, \preccurlyeq, \mathcal{T}\rangle$ such that $\langle X; \preccurlyeq, \mathcal{T}\rangle$ is a Priestley space and f is an order-reversing homeomorphism of order 2.

To prove that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard, let $\mathbf{X} = \langle C; f, \preccurlyeq, \mathcal{T}\rangle$ where $\langle C; \mathcal{T}\rangle$ is the Cantor space, $\preccurlyeq$ is the Stralka order and $f(x) = 1 - x$. Then $\mathbf{X}$ is locally finite, f is continuous and every finite substructure of $\mathbf{X}$ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$, but $\langle C; \preccurlyeq, \mathcal{T}\rangle$ is not a Priestley space. Thus $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard by Corollary 1.3. □

Example 2.12 *The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{K}}$, strongly dual to Kleene algebras, is non-standard.*

Proof. Davey and Werner [16] ([4]: 4.3.10) take $\underset{\sim}{\mathbf{K}} = \langle \{0, a, 1\}; \preccurlyeq, \sim, K_0, \mathcal{T}\rangle$ and define the order $\preccurlyeq$ with $0 \preccurlyeq a$ and $1 \preccurlyeq a$, the binary relation $\sim\, = \{0, a, 1\}^2 \setminus \{(0, 1), (1, 0)\}$, and the unary relation $K_0 = \{0, 1\}$. The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ consists of all Boolean structures $\mathbf{X} = \langle X; \preccurlyeq, \sim, X_0, \mathcal{T}\rangle$ such that $\langle X; \preccurlyeq, \mathcal{T}\rangle$ is a Priestley space and $\mathbf{X}$ satisfies the universal axioms

(a) $x \sim x$,
(b) $x \sim y$ and $x \in X_0 \implies x \preccurlyeq y$,
(c) $x \sim y$ and $y \preccurlyeq z \implies z \sim x$.

To establish that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard, let $\mathbf{X} = \langle C; \preccurlyeq, \sim, X_0, \mathcal{T}\rangle$ where $\langle C; \mathcal{T}\rangle$ is the Cantor space, $\preccurlyeq$ is the Stralka order, $\sim\, = \,\preccurlyeq \cup \succcurlyeq$ and $X_0 = \{0, 1\}$. Then $\sim$ and X_0 are closed so that $\mathbf{X}$ is a Boolean structure, and $\mathbf{X}$ is locally finite. We easily check that $\mathbf{X}$ satisfies the universal axioms (a), (b) and (c) and consequently so does every substructure of $\mathbf{X}$. Thus every finite substructure of $\mathbf{X}$ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. Since the space $\langle C; \preccurlyeq, \mathcal{T}\rangle$ is not a Priestley space, $\mathbf{X} \notin \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ and therefore $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard by Corollary 1.3. □

Example 2.13 *The category* $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$, *strongly dual to median algebras, is non-standard.*

Proof. By Isbell [20] and Werner [29] ([4]: 4.3.4) we let $\underset{\sim}{\mathbf{M}} = \langle\{0,1\}; {}^*, 0, 1, \leqslant, \mathcal{T}\rangle$ where $\leqslant$ is the usual order and * interchanges 0 and 1. The category $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ consists of all Boolean structures $\mathbf{X} = \langle X; {}^*, 0, 1, \leqslant, \mathcal{T}\rangle$ such that $\langle X; 0, 1, \leqslant, \mathcal{T}\rangle$ is a bounded Priestley space, * is order-reversing, interchanges 0 and 1, and satisfies $x^{**} \approx x$ and $x \leqslant x^* \implies x \approx 0$.

To verify that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard, let $\mathbf{X} = \langle C; {}^*, 0, 1, \leqslant, \mathcal{T}\rangle$ where $\langle C; \mathcal{T}\rangle$ is the Cantor space, $\leqslant$ is the Stralka order with the pair $(\frac{1}{3}, \frac{2}{3})$ removed, and $x^* = 1-x$. Clearly $\leqslant$ is closed, * is continuous, and $\mathbf{X}$ is locally finite. Moreover $\mathbf{X}$ satisfies the required universal axioms and consequently every finite substructure of $\mathbf{X}$ is in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. But the space $\langle C; \leqslant, \mathcal{T}\rangle$ is again not a Priestley space so $\mathbf{X} \notin \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$. Thus $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})$ is non-standard by Corollary 1.3. □

3. Misguided conjectures

Given a finite structure $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$, we would like to know whether or not its generated topological quasi-variety $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_c\mathbb{P}^+\,\underset{\sim}{\mathbf{M}}$ is standard. The examples in the previous section all arose as natural duals of familiar quasi-varieties. As a result, information was available about them that allowed us to answer this question in those cases. But we should bear in mind that the question of standardness applies to any choice of $\underset{\sim}{\mathbf{M}}$, regardless of whether or not it is associated with a natural duality. Based on the examples we have seen, we might venture a natural guess.

Misguided Conjecture 3.1 *A finite structure* $\underset{\sim}{\mathbf{M}} = \langle M; G, H, R, \mathcal{T}\rangle$ *is standard provided that it is a total or partial algebra, that is,* $R = \varnothing$.

Let $\underline{\mathbf{M}} = \langle\{0, a, 1\}; \vee, \wedge, 0, 1\rangle$ be the three-element chain. Then $\underline{\mathbf{M}}$ generates the (quasi-)variety of distributive lattices and has two proper endomorphisms, f and g, where $f(a) = 0$ and $g(a) = 1$. In [10] Davey, Haviar and Priestley show that $\underset{\sim}{\mathbf{M}}_{fg} = \langle\{0, a, 1\}; f, g, \mathcal{T}\rangle$ yields a duality on bounded distributive lattices relative to the generator $\underline{\mathbf{M}}$, and Davey and Haviar [8] verified that this duality is neither full nor strong.

Counterexample 3.2 *The total algebra* $\underset{\sim}{\mathbf{M}}_{fg} = \langle \{0, a, 1\}; f, g, \mathcal{T} \rangle$ *generates a non-standard topological quasi-variety* $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fg}$.

Proof. Given the set Z of all integers, the structure $\mathbf{Z}_\infty = \langle Z \cup \{\infty\}; f, g, \mathcal{T} \rangle$ is defined as follows. (See Figure 1.) For all $n \in Z$, let $f(2n) = g(2n) = 2n$, let $f(2n+1) = 2n$, let $g(2n+1) = 2n+2$ and let $f(\infty) = g(\infty) = \infty$. Topologically $\mathbf{Z}_\infty$ is the one point compactification of Z, where $U \subseteq Z \cup \{\infty\}$ is open if it is either cofinite or does not contain ∞. It is easy to check that $\mathbf{Z}_\infty$ is a Boolean structure and that every finite substructure of $\mathbf{Z}_\infty$ can be separated by morphisms into $\underset{\sim}{\mathbf{M}}_{fg}$ and is therefore in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fg}$. It is also easy to check that the only continuous homomorphisms from $\mathbf{Z}_\infty$ into $\underset{\sim}{\mathbf{M}}_{fg}$ are the constant maps to 0 and to 1, and consequently $\mathbf{Z}_\infty$ is not in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fg}$. By Corollary 1.3 we conclude that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fg}$ is non-standard. □

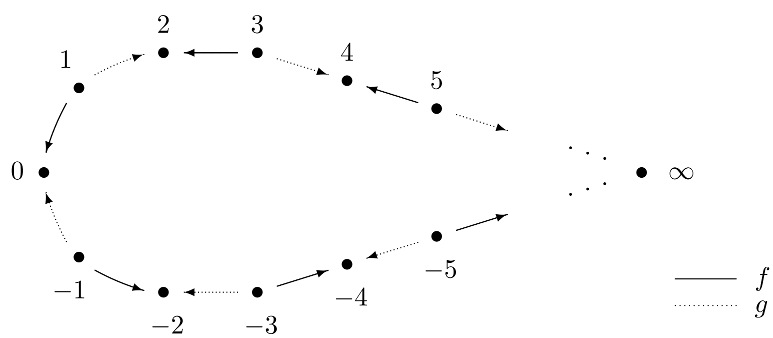

Figure 1. $\mathbf{Z}_\infty$

In fact $\underset{\sim}{\mathbf{M}}_{fg}$ has a special property that might possibly account for this aberrant behavior.

Lemma 3.3 *The quasi-equational theory* $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_{fg})$ *is not finitely axiomatizable.*

Proof. Let $\Sigma \subseteq \mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_{fg})$ be finite, and let n be the number of variables that occur in Σ. Let $\mathbf{Y} = \langle \{0, 1, \ldots, 2n+1\}; f, g, \mathcal{T} \rangle$ be the discrete $(2n+2)$-cycle where

$$0 \overset{f}{\longleftarrow} 1 \overset{g}{\Longrightarrow} 2 \overset{f}{\longleftarrow} 3 \overset{g}{\Longrightarrow} 4 \overset{f}{\longleftarrow} \ldots \overset{g}{\Longrightarrow} 2n \overset{f}{\longleftarrow} 2n+1 \overset{g}{\Longrightarrow} 0$$

and $f(2k) = g(2k) = 2k$ for all $k \leqslant n$. It is easy to check that the only (continuous) homomorphisms from $\mathbf{Y}$ into $\underset{\sim}{\mathbf{M}}_{fg}$ are the constant maps to 0 and to 1, and consequently $\mathbf{Y}$ is not in $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fg}$. By Lemma 1.1 we see that $\mathbf{Y}$ is not a model of $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_{fg})$.

On the other hand it is also easy to check that every proper subalgebra $\mathbf{X}$ of $\mathbf{Y}$ can be separated by homomorphisms into $\underset{\sim}{\mathbf{M}}_{fg}$. By the Separation Theorem we have $\mathbf{X} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fg}$, and so, by the Preservation Theorem, we conclude that $\mathbf{X}$ is a model of Σ. Since every subset of $\mathbf{Y}$ with no more than n elements generates a proper subalgebra of $\mathbf{Y}$, we see that $\mathbf{Y}$ is a model of Σ. Thus Σ does not axiomatize $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_{fg})$. □

The previous lemma also follows from [2], by Bestsennyi, which contains a complete description of the three-element unary algebras whose quasi-equational theory is finitely axiomatizable.

Misguided Conjecture 3.4 *A finite structure $\underset{\sim}{\mathbf{M}}$ is standard provided that $R = \varnothing$ and that its quasi-equational theory $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$ is finitely axiomatizable.*

We shall now construct a counterexample to this conjecture. Again, let $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, 0, 1 \rangle$ be the three-element chain. Within the square $\underline{\mathbf{M}} \times \underline{\mathbf{M}}$, the set $\{(0,0), (0,1), (1,1)\}$ forms a bounded sublattice $\mathbf{N}$ isomorphic to $\underline{\mathbf{M}}$. We will view the isomorphism $m : \mathbf{N} \to \underline{\mathbf{M}}$ as a binary partial operation on the set $\{0, a, 1\}$. Let $\underset{\sim}{\mathbf{M}}_{fgm} = \langle \{0, a, 1\}; f, g, m, \mathcal{T} \rangle$, be the partial algebra obtained by adding m to $\underset{\sim}{\mathbf{M}}_{fg}$, and let $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fgm} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}^{+}\,\underset{\sim}{\mathbf{M}}_{fgm}$ be the topological quasi-variety it generates.

Lemma 3.5 *The partial algebra $\underset{\sim}{\mathbf{M}}_{fgm}$ generates a non-standard topological quasi-variety $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fgm}$.*

Proof. We form a Stralka-based Boolean structure $\mathbf{Z} = \langle Z; f, g, m, \mathcal{T} \rangle$ in several steps. Let $\langle C; \leqslant, \mathcal{T} \rangle$ be the Cantor space with the Stralka order of Example 2.8. Now add to C the center point of each missing third. These are exactly the numbers between 0 and 1 whose infinite decimal expansion, base 3, consists of a finite sequence of 0s and 2s followed by an infinite sequence of 1s. Finally, bend the space into a circle by identifying 0 and 1 as a single point Q. Let C' be the image of C under this construction. We define $\langle Z; \leqslant, \mathcal{T} \rangle$ to be the resulting Boolean space with the quotient topology and the Stralka order $\leqslant$ on C', where now $Q \leqslant Q$. Let $f : Z \to Z$ fix each point of C' and map each new center point to the point of C' immediately below it in the usual order. Similarly, let $g : Z \to Z$

fix each point of C' and map each new center point to the point of C' immediately above it in the usual order. Finally, let m be the binary partial operation on Z whose domain is $\leqslant$. We take $m(x,y)$ to be x if $x = y$ and to be the midpoint of the interval $[x,y]$ if $x < y$. The following facts about the structure $\mathbf{Z}$ are now straightforward to verify.

- $\mathbf{Z}$ is a Boolean structure, that is, a Boolean space with $\leqslant$ closed and f, g and m continuous.
- The only morphisms from $\mathbf{Z}$ to $\underset{\sim}{\mathbf{M}}_{fgm}$ are the constant maps to 0 and to 1. Thus $\mathbf{Z} \notin \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fgm}$.
- $\mathbf{Z}$ is locally finite.
- For every finite substructure $\mathbf{Y}$ of $\mathbf{Z}$, there are separating morphisms from $\mathbf{Y}$ into $\underset{\sim}{\mathbf{M}}_{fgm}$. Thus $\mathbf{Y} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fgm}$.

It follows from Corollary 1.3 that $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fgm}$ is non-standard. □

Lemma 3.6 *The quasi-equational theory* $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_{fgm})$ *is finitely axiomatizable.*

Proof. We take Σ to be the following (admittedly redundant) set of axioms, all of which hold in $\underset{\sim}{\mathbf{M}}_{fgm}$.

A1) $ff(x) \approx f(x) \approx gf(x),\ \ gg(x) \approx g(x) \approx fg(x)$
A2) $m(x,y) \approx m(x,y) \implies f(m(x,y)) \approx x \ \&\ g(m(x,y)) \approx y$
A3) $m(x,x) \approx m(x,x) \iff f(x) \approx x$
A4) $m(x,y) \approx m(y,x) \implies x \approx y$
A5) $m(x,y) \approx m(x,y) \ \&\ m(y,z) \approx m(y,z) \implies m(x,z) \approx m(x,z)$
A6) $m(f(x), g(x)) \approx x$
A7) $f(x) \approx g(x) \implies f(x) \approx x$
A8) $m(m(x,y),u) \approx m(m(x,y),u) \implies m(x,y) \approx x$
A9) $m(x, m(u,v)) \approx m(x, m(u,v)) \implies m(u,v) \approx u$

Recall that, in the presence of partial operations, an equation is satisfied if both sides are defined and are equal. Thus, for example, the odd-looking equation $m(m(x,y),u) \approx m(m(x,y),u)$ simply asserts that both $(x,y) \in \mathrm{dom}(m)$ and $(m(x,y),u) \in \mathrm{dom}(m)$.

Let $\mathbf{X} = \langle X; f, g, m, \mathcal{T}\rangle \models \Sigma$. We will prove that $\mathbf{X} \models \mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}_{fgm})$ using Lemma 1.1. We first argue that $\mathbf{X}$ is locally finite. Let $Z \subseteq X$ be finite. Then $W := Z \cup f(Z) \cup g(Z)$ is a finite set closed under f and g by A1. Now define $Y := W \cup m(W^2 \cap \mathrm{dom}(m))$. Then Y is a finite subset of X containing Z, and Y is closed under f and g by A2 and is closed under m by A8 and A9.

Now let $\mathbf{Y}$ be a finite substructure of $\mathbf{X}$. Then $\mathbf{Y} \models \Sigma$. By A1 we see that f and g are both retractions of Y onto the same set $Y_0 \subseteq Y$. From A3, A4 and A5, we know that the domain of m is an order on Y_0 which we will denote by $\leqslant$.

In order to produce separating homomorphisms into $\underset{\sim}{\mathbf{M}}_{fgm}$, let U be a $\leqslant$–increasing subset of Y_0. For $x \in Y$ we have $f(x) \leqslant g(x)$ by A6, so we can define a map $\alpha_U : Y \to \{0, a, 1\}$ by

$$\alpha_U(x) = \begin{cases} 1 & \text{if } f(x) \in U \text{ (and hence } g(x) \in U), \\ a & \text{if } f(x) \notin U \text{ and } g(x) \in U, \\ 0 & \text{if } g(x) \notin U \text{ (and hence } f(x) \notin U). \end{cases}$$

We claim that $\alpha_U : \mathbf{Y} \to \underset{\sim}{\mathbf{M}}_{fgm}$ is a morphism. Since $\mathbf{Y}$ is finite, α_U is continuous. To see that α_U preserves f, let $x \in Y$. By A1 we have $ff(x) = gf(x)$. If $f(x) \in U$, then $\alpha_U(x) = 1$ so $\alpha_U(f(x)) = 1 = f(\alpha_U(x))$. If $f(x) \notin U$, then $\alpha_U(x) = 0$ or $\alpha_U(x) = a$ so (in both cases) $\alpha_U(f(x)) = 0 = f(\alpha_U(x))$. Thus α_U preserves f, and similarly it preserves g.

The fact that α_U preserves m follows from A2, A6 and the fact that it preserves f and g. To see this, let $x, y, z \in Y$ with $m(x, y) = z$. By A2 we have $f(z) = x$ and $g(z) = y$, and therefore $f(\alpha_U(z)) = \alpha_U(x)$ and $g(\alpha_U(z)) = \alpha_U(y)$. Applying A6 we obtain $m(\alpha_U(x), \alpha_U(y)) = m(f(\alpha_U(z)), g(\alpha_U(z))) = \alpha_U(z) = \alpha_U(m(x, y))$.

We now use the maps α_U to verify the conditions of the Separation Theorem.

- (i) Assume $x, y \in Y$ where $x \neq y$. By A6, one of the sets $\{f(x), f(y)\}$ or $\{g(x), g(y)\}$ has two different members. Let U be a $\leqslant$–increasing subset of Y_0 splitting one of these sets. Then $\alpha_U(x) \neq \alpha_U(y)$.
- (ii) Assume $(x, y) \notin \mathrm{dom}(m)$. First assume that x and y are both in Y_0. Then we must have $x \nleqslant y$. Let $U = \{z \in Y_0 \mid x \leqslant z\}$. Then $(\alpha_U(x), \alpha_U(y)) = (1, 0) \notin \mathrm{dom}(m)$. Now assume x and y are not both in Y_0; say $x \notin Y_0$. Then $f(x) \neq g(x)$ by A7. As $f(x) \leqslant g(x)$ by A6, the set $U = \{z \in Y_0 \mid g(x) \leqslant z\}$ contains $g(x)$ but not $f(x)$. Thus $\alpha_U(x) = a$ and so $(\alpha_U(x), \alpha_U(y)) \notin \mathrm{dom}(m)$.

It follows that $\mathbf{Y} \in \mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}})_{fgm}$. □

Combining the previous two lemmas gives our counterexample to Misguided Conjecture 3.4.

Counterexample 3.7 *The partial algebra* $\underset{\sim}{\mathbf{M}}_{fgm} = \langle \{0, a, 1\}; f, g, m, \mathcal{T} \rangle$ *has a finitely axiomatizable quasi-equational theory yet generates a non-standard topological quasi-variety.*

Making one last attempt at a valid conjecture, we notice that the standard examples we saw in Section 2 all arose as strong duals of algebraic quasi-varieties.

Misguided Conjecture 3.8 *A finite structure $\underset{\sim}{\mathbf{M}}$ is standard provided that $R = \varnothing$, that its quasi-equational theory $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$ is finitely axiomatizable and that it strongly dualises some algebraic quasi-variety.*

In fact $\underset{\sim}{\mathbf{M}}_{fgm}$ was constructed in Davey and Haviar [8] for the explicit purpose of proving and generalizing the following fact, which the authors extend to a general method of transferring a strong duality to a different generator of the quasi-variety.

Counterexample 3.9 *The partial algebra $\underset{\sim}{\mathbf{M}}_{fgm} = \langle \{0, a, 1\}; f, g, m, \mathcal{T}\,\rangle$ yields a strong duality on the quasi-variety of bounded distributive lattices relative to the generator $\underline{\mathbf{M}} = \langle \{0, a, 1\}; \vee, \wedge, 0, 1\rangle$, the structure $\underset{\sim}{\mathbf{M}}_{fgm}$ has a finitely axiomatizable quasi-equational theory, and yet $\underset{\sim}{\mathbf{M}}_{fgm}$ generates a non-standard topological quasi-variety.*

Without risking any further conjectures, we conclude this section with two open problems.

Problem 3.10 *If $\underset{\sim}{\mathbf{M}}$ is a total algebra strongly dualising some algebra $\underline{\mathbf{M}}$, must it be standard?*

Problem 3.11 *If $\underset{\sim}{\mathbf{M}}$ is a total algebra whose quasi-equational theory has a finite basis, must it be standard?*

4. Every finite Boolean unar is standard.

Humbled by the experiences of the last section, we proceed with a healthy respect for the difficulty of the standardness problem. In this section we use Lemma 1.5 to exhibit a large class of finite structures which generate standard topological quasi-varieties. A *Boolean unar* is a Boolean structure $\mathbf{X} = \langle X; f, \mathcal{T}\,\rangle$ having a single unary operation f. A k–element subset $\{x_0, x_1, \dots, x_{k-1}\}$ of X is a *k–loop* if $f(x_i) = x_{i+1 (\mathrm{mod}\ k)}$ for each $i < k$. Let $L_{\mathbf{X}}$ denote the union of the loops of $\mathbf{X}$, which determines a substructure $\mathbf{L}_{\mathbf{X}}$ of $\mathbf{X}$. A *tail* of length l of $\mathbf{X}$ is an l–element subset $\{y_0, y_1, \dots, y_{l-1}\}$ of $X \setminus L_{\mathbf{X}}$ such that $f(y_j) = y_{j+1}$ for each $j < l - 1$. Finally, we define a *component of* $\mathbf{X}$ to be a substructure $\mathbf{C}$ of $\mathbf{X}$ such that $C = \bigcup \{\, f^{-n}(L) \mid n \in \mathbb{N}\,\}$, for some loop L of $\mathbf{X}$.

Lemma 4.1 *Let $\mathbf{X} = \langle X; f, \mathcal{T} \rangle$ be a Boolean unar satisfying $f^{2m}(x) \approx f^m(x)$, where $m \geqslant 1$. If $x, y \in X$ and $x \neq y$, then there is a finite substructure $\mathbf{Y}$ of $\mathbf{X}$ and a morphism $\alpha : \mathbf{X} \to \mathbf{Y}$ separating x and y.*

PROOF. We break the proof into a series of claims. The first claim will be used constantly throughout the proof without reference.

`Claim 1` *The map $f^m : \mathbf{X} \to \mathbf{X}$ is a retraction onto the loops of $\mathbf{X}$.*

PROOF. Note that $f^m : \mathbf{X} \to \mathbf{X}$ is a morphism, since f is continuous and preserves itself. As $\mathbf{X} \models f^{2m}(x) \approx f^m(x)$, we have $f^m(X) \subseteq L_{\mathbf{X}}$. Now let $x \in L_{\mathbf{X}}$. Then $f^k(x) = x$ for some $k \geqslant 1$. We have $x = (f^k)^m(x) = (f^m)^k(x) \in f^m(X)$. So $L_{\mathbf{X}} = f^m(X)$. To see that f^m is the identity on $L_{\mathbf{X}}$, let $x \in L_{\mathbf{X}}$. Then $x = f^m(y)$ for some $y \in X$, and we have $f^m(x) = f^{2m}(y) = f^m(y) = x$. $\square$

`Claim 2` *Let $\mathbf{C}$ be a component of $\mathbf{X}$ whose loop $L_{\mathbf{C}}$ has q elements. Then there are disjoint clopen subsets $U_0, U_1, \dots, U_{q-1}$ of $\mathbf{X}$, covering C, such that $f^{-1}(U_i) = U_{i+1 (\mathrm{mod}\ q)}$ for $i < q$. Furthermore, if $\mathbf{D}$ is another component of $\mathbf{X}$, then the sets $U_0, U_1, \dots, U_{q-1}$ can be chosen to be disjoint from D.*

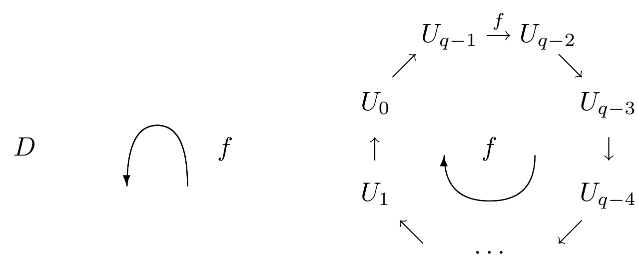

FIGURE 2. clopen cover of a component

PROOF. Choose some $z \in L_{\mathbf{C}}$. Since $\mathbf{X}$ is Boolean and $L_{\mathbf{C}}$ is finite, there is a clopen subset U of $\mathbf{X}$ such that $U \cap L_{\mathbf{C}} = \{z\}$. (If $\mathbf{D}$ is another component of $\mathbf{X}$, then we can also ensure that $U \cap L_{\mathbf{D}} = \varnothing$.) Define

$$U_0 := \{\, x \in X \mid f^{m+l}(x) \in U \text{ if and only if } q|l, \text{ for all } l \geqslant 0 \,\}.$$

Since $\mathbf{X}$ satisfies $f^{2m}(x) \approx f^m(x)$, we have $q|m$ and

$$U_0 = \bigcap\{\, f^{-(m+l)}(U) \mid l \in \{0,\dots,m\} \text{ and } q|l \,\} \Big\backslash \bigcup\{\, f^{-(m+l)}(U) \mid l \in \{0,\dots,m\} \text{ and } q\nmid l \,\}.$$

So U_0 is clopen, as f is continuous. Define the clopen set $U_i := f^{-i}(U_0)$, for each $0 < i < q$.

We first show that the sets $U_0, U_1, \dots, U_{q-1}$ cover C. Since $U \cap L_{\mathbf{C}} = \{z\}$ and $|L_{\mathbf{C}}| = q$, we have $f^l(z) \in U$ if and only if $q|l$, for all $l \geqslant 0$. This implies that $f^{-m}(z) \subseteq U_0$. Now let $x \in C$. We have $f^m(x) \in L_{\mathbf{C}}$, and so $f^{m+i}(x) = z$, for some $i < q$. This gives us $f^i(x) \in f^{-m}(z) \subseteq U_0$, and therefore $x \in f^{-i}(U_0) = U_i$. Thus C is covered by $U_0, U_1, \dots, U_{q-1}$.

Clearly, we have $f^{-1}(U_i) = U_{i+1}$, for all $i < q-1$. We want to show that $f^{-1}(U_{q-1}) = U_0$. First, let $x \in f^{-1}(U_{q-1})$. Then $f(x) \in U_{q-1}$ and so $f^q(x) = f^{q-1}(f(x)) \in U_0$. It follows that, for all $l \geqslant 0$, we have

$$f^{m+l}(x) \in U \iff f^{2m+l-q}(f^q(x)) \in U \iff q|(m+l-q) \iff q|l,$$

the first equivalence being true because $f^m(f^l(x)) = f^{2m}(f^l(x))$, the second because $f^q(x) \in U_0$ and the third because $q|m-q$. Thus $x \in U_0$, whence $f^{-1}(U_{q-1}) \subseteq U_0$. Now let $y \in U_0$. We want to check that $f(y) \in U_{q-1} = f^{-(q-1)}(U_0)$. For all $l \geqslant 0$, we have

$$f^{m+l}(f^{q-1}(f(y))) \in U \iff f^{m+l+q}(y) \in U \iff q|(l+q) \iff q|l.$$

So $f^{q-1}(f(y)) \in U_0$ and therefore $f(y) \in U_{q-1}$. Thus $f^{-1}(U_{q-1}) = U_0$.

We shall now show that the sets $U_0, U_1, \dots, U_{q-1}$ are disjoint. Assume that $x \in U_i \cap U_j$ for some $i \leqslant j < q$. Then $f^j(x) \in U_0$ and therefore $f^m(f^j(x)) \in U$, as $q|0$. This gives us

$$f^{m+j-i}(f^i(x)) = f^{m+j}(x) \in U.$$

Since $f^i(x) \in U_0$, we have $q|(j-i)$ and therefore $i = j$. Thus $U_0, U_1, \dots, U_{q-1}$ are pairwise disjoint.

It remains to show that, if we are working with the extra component $\mathbf{D}$, the sets $U_0, U_1, \dots, U_{q-1}$ are disjoint from D. Suppose that $x \in U_i \cap D$, for some $i < q$. Then $f^i(x) \in U_0$ and so $f^{m+i}(x) \in U$. But $f^{m+i}(x) \in f^m(D) = L_{\mathbf{D}}$ which contradicts the fact that $U \cap L_{\mathbf{D}} = \varnothing$. □

`Claim 3` *Let $\mathbf{C}$ be a component of $\mathbf{X}$ whose loop $L_{\mathbf{C}}$ has q elements and whose longest tail has length t. Let $x, y \in C$ with $x \neq y$ and $f^m(x) = f^m(y)$. For some*

$d < t$, there are disjoint clopen subsets $U_0, U_1, \ldots, U_{q-1}, T_0, T_1, \ldots, T_d$ of $\mathbf{X}$, covering C and separating x and y, such that

$$f^{-1}(T_d) = \varnothing, \quad f^{-1}(T_j) = T_{j+1}, \quad f^{-1}(U_{q-1}) = T_0 \cup U_0, \quad \text{and} \quad f^{-1}(U_i) = U_{i+1}$$

for $i < q-1$ and $j < d$.

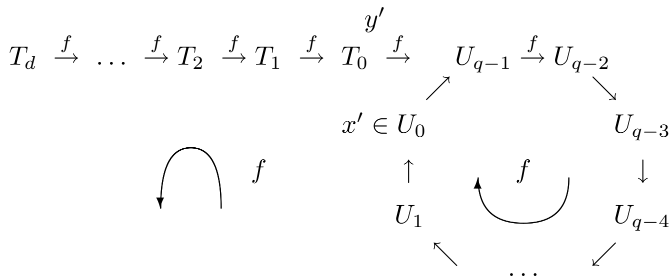

FIGURE 3. clopen cover of a component with separated tail

PROOF. Let $U_0', U_1', \ldots, U_{q-1}'$ be disjoint clopen sets covering C as in Claim 2. Define

$$n := \max\{\, l \geqslant 0 \mid f^l(x) \neq f^l(y) \,\}.$$

We shall separate $x' := f^n(x)$ and $y' := f^n(y)$. We may assume that $x', y' \in U_0'$, since $f(x') = f(y')$. We can also assume that $y' \notin L_{\mathbf{C}}$. Define

$$d := \max\{\, l \geqslant 0 \mid f^{-l}(y') \neq \varnothing \,\}.$$

Then $d < t \leqslant m$.

Since $\mathbf{X}$ is Boolean, there is a clopen set U, with $U \subseteq U_0'$, such that $y' \in U$ and $x' \notin U$. For each $z \in U \setminus \{y'\}$, let $U_z \subseteq U$ be a clopen set with $z \in U_z$ and $y' \notin U_z$. Now define $V := f^{-(d+1)}(U)$ and, for each $z \in U \setminus \{y'\}$, define $V_z := f^{-(d+1)}(U_z)$. Then $\{\, V_z \mid z \in U \setminus \{y'\} \,\}$ covers V, since $f^{-(d+1)}(y') = \varnothing$. As $\mathbf{X}$ is compact, there is a finite subset U_{fin} of $U \setminus \{y'\}$ such that $\{\, V_z \mid z \in U_{\text{fin}} \,\}$ covers V. So we can define the clopen subset

$$T_0' := U \setminus \left(\bigcup \{\, U_z \mid z \in U_{\text{fin}} \,\} \right)$$

of U_0' which contains y' but not x'.

We argue that $f^{-(d+1)}(T_0') = \varnothing$. Suppose that $w \in f^{-(d+1)}(T_0')$. Since $T_0' \subseteq U$, we have $w \in V$ and so $w \in V_z$, for some $z \in U_{\text{fin}}$. This implies that $f^{d+1}(w) \in U_z \cap T_0' = \varnothing$, which is a contradiction.

For $0 < j \le d$, define $T_j' := f^{-j}(T_0')$. We observe that y' is not in T_j' if $0 < j \le d$. For suppose that it were. From the choice of d, there is a $t \in T_d'$ with $f^d(t) = y'$. Then we would have

$$f^{d+1}(f^{j-1}(t)) = f^j(f^d(t)) = f^j(y') \in T_0'$$

since $y' \in T_j'$. Thus $f^{j-1}(t) \in f^{-(d+1)}(T_0') = \varnothing$, a contradiction.

We can therefore find a clopen set $T_0 \subseteq T_0'$ containing y' but not intersecting any T_j' with $0 < j \le d$. Define $T_j := f^{-j}(T_0)$ for $0 < j \le d$, and define $U_i := U_i' \setminus (T_0 \cup \cdots \cup T_d)$ for $0 \le i < q$.

To see that $U_0, U_1, \dots, U_{q-1}, T_0, T_1, \dots, T_d$ are disjoint, we assume that $t \in T_i \cap T_{i+j}$ with $j > 0$. Then $f^i(t)$ would be in $T_0 \cap T_j \subseteq T_0 \cap T_j'$ which is empty. Also, $f^{-1}(T_d)$ is empty as we have $f^{-1}(T_d) = f^{-(d+1)}(T_0) \subseteq f^{-(d+1)}(T_0') = \varnothing$. It is now straightforward to check that $f^{-1}(U_{q-1}) = T_0 \cup U_0$, that $f^{-1}(U_i) = U_{i+1}$ for $i < q-1$, and that $f^{-1}(T_j) = T_{j+1}$ for $j < d$. □

`Claim 4` *Let* $\mathbf{Z}$ *be a clopen substructure of* $\mathbf{X}$. *Then there is a morphism* $\varphi : \mathbf{Z} \to \mathbf{Z}$ *such that* $\varphi(Z)$ *is finite.*

Proof. For each component $\mathbf{C}$ of $\mathbf{Z}$, let $U_0^{\mathbf{C}}, U_1^{\mathbf{C}}, \dots, U_{q_{\mathbf{C}}-1}^{\mathbf{C}}$ be disjoint clopen subsets of $\mathbf{Z}$ which cover C, as in Claim 2, and define

$$U_{\mathbf{C}} := U_0^{\mathbf{C}} \cup U_1^{\mathbf{C}} \cup \cdots \cup U_{q_{\mathbf{C}}-1}^{\mathbf{C}}.$$

Since the space $\mathbf{Z}$ is compact, there is a finite set F of components of $\mathbf{Z}$ such that $\{ U_{\mathbf{C}} \mid \mathbf{C} \in F \}$ covers Z. Let $F = \{\mathbf{C}_1, \dots, \mathbf{C}_n\}$. We want to define a morphism $\varphi : \mathbf{Z} \to \mathbf{Z}$ such that

$$\varphi(U_{\mathbf{C}_1}) \subseteq L_{\mathbf{C}_1}, \quad \varphi(U_{\mathbf{C}_2} \backslash U_{\mathbf{C}_1}) \subseteq L_{\mathbf{C}_2}, \quad \dots \\ \dots, \quad \varphi(U_{\mathbf{C}_n} \backslash (U_{\mathbf{C}_1} \cup \cdots \cup U_{\mathbf{C}_{n-1}})) \subseteq L_{\mathbf{C}_n}.$$

To see that this is possible, let $i \in \{1, \dots, n\}$. For each $j \in \{1, \dots, n\}$, the set $U_{\mathbf{C}_j}$ is closed under both f and f^{-1} within $\mathbf{Z}$. So $V_i := U_{\mathbf{C}_i} \setminus (U_{\mathbf{C}_1} \cup \cdots \cup U_{\mathbf{C}_{i-1}})$ determines a clopen substructure of $\mathbf{Z}$. The clopen sets

$$V_i \cap U_0^{\mathbf{C}_i}, \quad V_i \cap U_1^{\mathbf{C}_i}, \quad \dots, \quad V_i \cap U_{q_{\mathbf{C}_i}-1}^{\mathbf{C}_i}$$

partition V_i and, for all $j \in \{0, \dots, q_{\mathbf{C}_i} - 1\}$, we have

$$f(V_i \cap U_j^{\mathbf{C}_i}) = V_i \cap U_{j'}^{\mathbf{C}_i},$$

where $j' = j - 1 \pmod{q_{\mathbf{C}_i}}$. So there is a morphism $\varphi_i : \mathbf{V}_i \to \mathbf{L}_{\mathbf{C}_i}$.

The set $\{\mathbf{V}_1, \ldots, \mathbf{V}_n\}$ of clopen substructures partitions $\mathbf{Z}$. Therefore we can define the morphism $\varphi : \mathbf{Z} \to \mathbf{Z}$ by $\varphi := \varphi_1 \cup \cdots \cup \varphi_n$. The range of φ is the finite set $L_{\mathbf{C}_1} \cup \cdots \cup L_{\mathbf{C}_n}$. □

We can now finish the proof of the lemma. Let $x, y \in X$ with $x \neq y$. Let $\mathbf{C}_x$ and $\mathbf{C}_y$ be the components of $\mathbf{X}$ containing x and y.

`Case (1):` $\mathbf{C}_x \neq \mathbf{C}_y$. Choose clopen subsets $U_0, U_1, \ldots, U_{q-1}$ of $\mathbf{X}$ which cover C_x and are disjoint from C_y, as in Claim 2. The sets $Z := U_0 \cup U_1 \cup \cdots \cup U_{q-1}$ and $Z' := X \setminus Z$ form clopen substructures $\mathbf{Z}$ and $\mathbf{Z}'$ of $\mathbf{X}$ with $C_x \subseteq Z$ and $C_y \subseteq Z'$. By Claim 4, there are morphisms $\varphi_x : \mathbf{Z} \to \mathbf{Z}$ and $\varphi_y : \mathbf{Z}' \to \mathbf{Z}'$ with finite ranges. So the morphism $\varphi := \varphi_x \cup \varphi_y : \mathbf{X} \to \mathbf{X}$ separates x and y and has a finite range.

`Case (2):` $\mathbf{C}_x = \mathbf{C}_y$ and $f^m(x) = f^m(y)$. Using Claim 3, we can find clopen sets $U_0, U_1, \ldots, U_{q-1}, T_0, T_1, \ldots, T_d$ that cover C_x and separate x and y. Define the sets $Z := U_0 \cup \cdots \cup U_{q-1} \cup T_0 \cup \cdots \cup T_d$ and $Z' := X \setminus Z$. Then Z and Z' determine clopen substructures $\mathbf{Z}$ and $\mathbf{Z}'$ of $\mathbf{X}$. The partition $\{U_0, \ldots, U_{q-1}, T_0, \ldots, T_d\}$ of Z determines a congruence of $\mathbf{Z}$. By construction, the loop of the component $\mathbf{C}_x$ has size q and there is a tail of $\mathbf{C}_x$ with length at least d. So the congruence $\{U_0, \ldots, U_{q-1}, T_0, \ldots, T_d\}$ is the kernel of a morphism $\varphi : \mathbf{Z} \to \mathbf{C}_x$. This morphism separates x and y and has a finite range. By Claim 4, we know that there is a morphism $\varphi' : \mathbf{Z}' \to \mathbf{Z}'$ that has finite range. Hence $\varphi \cup \varphi' : \mathbf{X} \to \mathbf{X}$ separates x and y and has a finite range.

`Case (3):` $\mathbf{C}_x = \mathbf{C}_y$ and $f^m(x) \neq f^m(y)$. This case is similar to Case 2, but Claim 2 should be used instead of Claim 3. (The sets $T_0, T_1, \ldots, T_d$ are not required and the function φ may now map into $\mathbf{L}_x$.) □

Theorem 4.2 *Every finite Boolean unar is standard.*

Proof. Consider a finite Boolean unar $\underset{\sim}{\mathbf{M}} = \langle M; f, \mathcal{T} \rangle$. Let n be the least common multiple of the sizes of the loops of $\underset{\sim}{\mathbf{M}}$, let t be the length of the longest tail in $\underset{\sim}{\mathbf{M}}$ and let $m = nt$. Notice that f^m maps $\underset{\sim}{\mathbf{M}}$ into $L_{\underset{\sim}{\mathbf{M}}}$ and fixes each element of $L_{\underset{\sim}{\mathbf{M}}}$. Thus $\underset{\sim}{\mathbf{M}}$ satisfies $f^{2m}(x) \approx f^m(x)$.

Now let $\mathbf{X} \in \mathrm{Mod}_{\mathcal{T}}(\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}}))$. Then $\mathbf{X}$ also satisfies $f^{2m}(x) \approx f^m(x)$. By Lemma 4.1 the conditions of Lemma 1.5 hold, and therefore $\underset{\sim}{\mathbf{M}}$ is standard.

□

5. Every finite Boolean unar is finitely axiomatizable.

Knowing in advance that a topological quasi-variety generated by a finite Boolean unar $\underset{\sim}{\mathbf{M}} = \langle M; f, \mathcal{T} \rangle$ is standard will now vastly simplify the axiomatization process. Applying Corollary 1.4 we will show that a particular finite set $\Sigma_{\underset{\sim}{\mathbf{M}}}$ axiomatizes $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$. Working only with finite structures, whose topology is discrete, we are now freed of the obligation to continue patching together clopen sets. Essentially, all we need to do is to find a finite quasi-equational basis for the unar as an algebra. Such a basis can be extracted from more general results in the paper [21], by Kartashov. In this section, we present a more direct construction of a basis.

Again we let n be the least common multiple of the sizes of the loops of $\underset{\sim}{\mathbf{M}}$, let t be the length of the longest tail in $\underset{\sim}{\mathbf{M}}$ and let $m = nt$. Our proof that $\underset{\sim}{\mathbf{M}}$ is standard used only the fact that $\underset{\sim}{\mathbf{M}} \models f^{2m}(x) \approx f^m(x)$. In general this single equation does not axiomatize $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$. For example, if $\underset{\sim}{\mathbf{M}}$ consists of a 6-loop and a 10-loop, then $\underset{\sim}{\mathbf{M}} \models f^{60}(x) \approx f^{30}(x) \approx x$. This equation is also satisfied by a 15-loop, but the loops of a power of $\underset{\sim}{\mathbf{M}}$ all have size 6, 10 or 30; not 15.

In order to exhibit a complete set of axioms for $\mathrm{Th}_{uH}(\underset{\sim}{\mathbf{M}})$, let Q be the set of positive integers that are the least common multiple (lcm) of the sizes of some non-empty set of loops of $\underset{\sim}{\mathbf{M}}$. Thus

$$Q \subseteq \{1, 2, \dots, n\}.$$

Notice that the lcm of a subset of Q is again in Q. Now let $a \in M$. The subalgebra of $\underset{\sim}{\mathbf{M}}$ generated by a is the disjoint union of a (possibly empty) tail, T_a, and a loop, L_a. We must have $f^t(a) \in L_a$. Since $|L_a|$ divides n, the loop element $f^t(a)$ is fixed by f^n. Thus $\underset{\sim}{\mathbf{M}}$ satisfies

$$f^{t+n}(x) \approx f^t(x). \tag{A}$$

For each $q \in Q$, let $t(q)$ be the length of the longest tail attached to a loop of $\underset{\sim}{\mathbf{M}}$ whose size divides q. Assume that $f^{t+q}(a) = f^t(a)$, for some $q \in Q$. Since $f^t(a) \in L_a$, this tells us that $|L_a|$ divides q. Thus $|T_a| \leqslant t(q)$, and therefore $f^{t(q)+q}(a) = f^{t(q)}(a)$. So $\underset{\sim}{\mathbf{M}}$ satisfies

$$f^{t+q}(x) \approx f^t(x) \implies f^{t(q)+q}(x) \approx f^{t(q)}(x) \tag{B$_q$}$$

for each $q \in Q$.

Now assume that $f^r(a) = a$, for some $r > 0$. Then r is a multiple of $|L_a|$ and $|L_a| \in Q$. Thus $\underset{\sim}{\mathbf{M}}$ satisfies

$$f^r(x) \not\approx x \tag{C$_r$}$$

for each positive integer r which is not a multiple of any member of Q.

On the other hand, assume that r is a multiple of some member of Q and let $s \in Q$ be the lcm of the members of Q which divide r. Again assume that $f^r(a) = a$. Then $|L_a|$ divides r. Since $|L_a| \in Q$, this implies that $|L_a|$ divides s. So $f^s(a) = a$. Thus $\underset{\sim}{\mathbf{M}}$ satisfies

$$f^r(x) \approx x \implies f^s(x) \approx x \tag{D_r}$$

where some member of Q divides r, and s is the lcm of all q in Q that divide r.

Finally, it may happen that $\underset{\sim}{\mathbf{M}}$ contains exactly one 1-element loop, which is expressed by the implication

$$f(x) \approx x \;\&\; f(y) \approx y \implies x \approx y. \tag{E}$$

We now define $\Sigma_{\underset{\sim}{\mathbf{M}}}$ to consist of

- (A),
- (B_q) for each $q \in Q$,
- (C_r) for each $r \leqslant n$ which is not a multiple of any member of Q,
- (D_r) for each $r \leqslant n$ which is a multiple of some member of Q, and
- (E) just in case $\underset{\sim}{\mathbf{M}}$ has exactly one 1-element loop.

Theorem 5.1 *If $\underset{\sim}{\mathbf{M}}$ is a finite Boolean unar, then $\Sigma_{\underset{\sim}{\mathbf{M}}}$ is a finite axiomatization for $\mathfrak{Q}_{\mathcal{T}}(\underset{\sim}{\mathbf{M}}) = \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$.*

Proof. We apply Corollary 1.4. Clearly every model of (A) is locally finite. Let $\mathbf{X} = \langle X; f, \mathcal{T}\rangle$ be a finite model of $\Sigma_{\underset{\sim}{\mathbf{M}}}$. Let L be a loop of $\mathbf{X}$ and define $r := |L|$.

We first observe that the size r of L is in Q. To see this, first note that $r = |L| \leqslant n$ by (A). Since $f^r(x) = x$, for all $x \in L$, the structure $\mathbf{X}$ does not satisfy (C_r). Thus $(\mathrm{C}_r) \notin \Sigma_{\underset{\sim}{\mathbf{M}}}$ and we conclude that r is a multiple of some member of Q. Let s be the lcm of all members of Q that divide r. Then $f^s(x) = x$, for all $x \in L$, by (D_r). So r divides s. But clearly s divides r, and therefore $r = s$ which is in Q.

Next we show that $\mathbf{L} = \langle L; f, \mathcal{T}\rangle$ can be embedded into a finite power of $\underset{\sim}{\mathbf{M}}$. Since $r \in Q$, there is some $k > 0$ such that $r = \mathrm{lcm}\{l_0, l_1, \dots, l_{k-1}\}$, where $m_i \in M$ generates an l_i–loop of $\underset{\sim}{\mathbf{M}}$ for each $i < k$. So $(m_0, m_1, \dots, m_{k-1}) \in M^k$ generates an r–loop of $\underset{\sim}{\mathbf{M}}^k$ isomorphic to $\mathbf{L}$.

We will now prove that $\mathbf{X} \in \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$ by showing that there is at least one morphism from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$ and then verifying condition (i) of the Separation Theorem. To produce one morphism, first note that $f^t : \mathbf{X} \to \mathbf{L}_{\mathbf{X}}$ by (A). Each loop of $\mathbf{X}$ can be embedded into a power of $\underset{\sim}{\mathbf{M}}$ which, in turn, can be mapped into $\underset{\sim}{\mathbf{M}}$. Using the fact that $\mathbf{L_X}$ is the coproduct of the loops of $\mathbf{X}$, it follows that there is at least one morphism from $\mathbf{X}$ into $\underset{\sim}{\mathbf{M}}$.

Now consider $x, y \in X$ where $x \neq y$. We want to construct a morphism from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}$ which separates x and y. Let $\mathbf{C}_x$ and $\mathbf{C}_y$ be the components of $\mathbf{X}$ containing x and y. We know that there is at least one morphism from $\mathbf{X}$ to $\underset{\sim}{\mathbf{M}}$, and that $\mathbf{X}$ is the coproduct of its components. So it is enough to find a morphism $\gamma : \mathbf{C}_x \cup \mathbf{C}_y \to \underset{\sim}{\mathbf{M}}$ separating x and y.

`Case (1):` $\mathbf{C}_x \neq \mathbf{C}_y$. First assume that $\underset{\sim}{\mathbf{M}}$ has two 1–loops. Then there is a morphism $\gamma : \mathbf{C}_x \cup \mathbf{C}_y \to \underset{\sim}{\mathbf{M}}$ which maps C_x onto one 1–loop of $\underset{\sim}{\mathbf{M}}$ and maps C_y onto another 1–loop of $\underset{\sim}{\mathbf{M}}$. This morphism will necessarily separate x and y.

We can now assume that $\underset{\sim}{\mathbf{M}}$ does not have two 1–loops. By (A), the morphism $f^t : \mathbf{X} \to \mathbf{L_X}$ maps C_x and C_y, respectively, into the loops L_x and L_y of $\mathbf{X}$. Since $\underset{\sim}{\mathbf{M}}$ does not have two 1–loops, either (C_1) or (E) will tell us that $\mathbf{L}_x$ and $\mathbf{L}_y$ cannot both be 1–loops of $\mathbf{X}$. Assume that $\mathbf{L}_y$ is not a 1–loop. Then $f^{t+1}(y) \neq f^t(y)$. Since $\mathbf{L}_x$ can be embedded into a power of $\underset{\sim}{\mathbf{M}}$, there is a morphism $\alpha : \mathbf{L}_x \to \underset{\sim}{\mathbf{M}}$. As $\mathbf{L}_y$ can be embedded into a power of $\underset{\sim}{\mathbf{M}}$, there is a morphism $\beta : \mathbf{L}_y \to \underset{\sim}{\mathbf{M}}$ with $\beta(f^{t+1}(y)) \neq \beta(f^t(y))$. Define the morphisms $\gamma, \delta : \mathbf{C}_x \cup \mathbf{C}_y \to \underset{\sim}{\mathbf{M}}$ by $\gamma := (\alpha \circ f^t){\restriction}_{C_x} \cup (\beta \circ f^{t+1}){\restriction}_{C_y}$ and $\delta := (\alpha \circ f^t){\restriction}_{C_x} \cup (\beta \circ f^t){\restriction}_{C_y}$. Then either γ or δ separates the elements x and y.

`Case (2):` $\mathbf{C}_x = \mathbf{C}_y$ and $f^t(x) \neq f^t(y)$. Embedding $\mathbf{L}_x$ into a power of $\underset{\sim}{\mathbf{M}}$ and then mapping into $\underset{\sim}{\mathbf{M}}$, there is a morphism $\alpha : \mathbf{L}_x \to \underset{\sim}{\mathbf{M}}$ which separates $f^t(x)$ and $f^t(y)$. Now $\alpha \circ f^t{\restriction}_{C_x} : \mathbf{C}_x \to \underset{\sim}{\mathbf{M}}$ separates x and y.

`Case (3):` $\mathbf{C}_x = \mathbf{C}_y$ and $f^t(x) = f^t(y)$. Define $C := C_x$ and $L := L_x$. Since $x \neq y$, it cannot be that both x and y belong to L. So we can assume that x is not an ancestor of y, that is, we can assume that $f^i(x) \neq y$, for all $i > 0$. This implies that y does not belong to the loop L. Let $L = \{x_0, x_1, \dots, x_{q-1}\}$, where $q := |L|$ and $f(x_i) = x_{i+1 (\mathrm{mod}\ q)}$ for all $i < q$. Since $y \notin L$, there is an integer s, with $0 < s \leqslant t$, such that $f^{-s}(y) = \varnothing$ but $f^{-(s-1)}(y) \neq \varnothing$. This gives us a partition

$$A_y := f^{-(s-1)}(y) \cup \cdots \cup f^{-2}(y) \cup f^{-1}(y) \cup \{y\}$$

of the set A_y of ancestors of y, where $f(f^{-j}(y)) \subseteq f^{-(j-1)}(y)$ for $1 \leqslant j < s$. Since $f^t{\restriction}_C : \mathbf{C} \to \mathbf{L}$ is a morphism and $f^t{\restriction}_L$ is one-to-one, the kernel of $f^t{\restriction}_C$ is a congruence on $\mathbf{C}$ given by the partition

$$C = f^{-t}(x_0) \cup f^{-t}(x_1) \cup \cdots \cup f^{-t}(x_{q-1}),$$

where

$$f(f^{-t}(x_i)) \subseteq f^{-t}(x_{i+1 (\mathrm{mod}\ q)})$$

for all $i < q$. Let $C_i := f^{-t}(x_i) \setminus A_y$ for each $i < q$. Neither $f(x)$ nor $f(y)$ is in A_y, since we are assuming that x is not an ancestor of y. As $f^t(f(x)) = f^t(f(y))$, we may assume that $f(x), f(y) \in C_1$. Then $x \in C_0$. This gives us a refined partition of C which forms the congruence shown in Figure 4. Using this congruence, we can separate x and y by a morphism from $\mathbf{C}$ into $\underset{\sim}{\mathbf{M}}$ provided that we can find within $\underset{\sim}{\mathbf{M}}$ an element m such that $|T_m| \geqslant s$ and $|L_m|$ divides q.

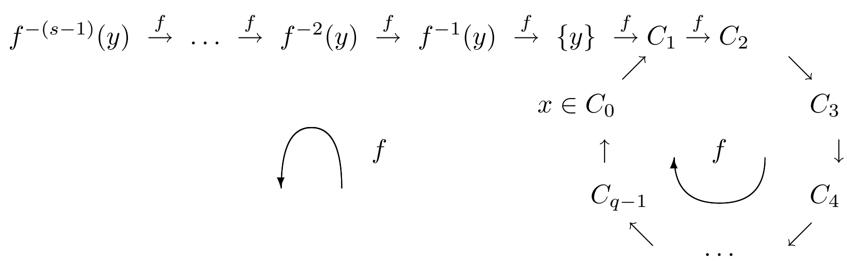

FIGURE 4. a congruence on the component $\mathbf{C}$

Now $|T_y| \geqslant s$ and $L_y = L$ has size q. We now apply (B_q), which says that if an element has a loop whose size divides q, then its tail has length at most $t(q)$. Thus $t(q) \geqslant |T_y| \geqslant s$. From the definition of $t(q)$, there is an element $m \in M$ such that $|T_m| = t(q) \geqslant s$ and $|L_m|$ divides q. This guarantees that x and y can be separated by a morphism from $\mathbf{C}$ into $\underset{\sim}{\mathbf{M}}$. □

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Received April 15, 2002

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