G.S. WATSON lecture

MATHEMATICS OF OPTIONS

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Options theory has changed financial markets and some say even capitalism itself.

The talk is about the main mathematical ideas in option theory.

1. Background

The risk in the stock market is in price fluctuation of shares. If you buy shares and prices fall you lose money.

An option allows you to reduce the risk and buy the stock at the predetermined price. If the price of the stock falls, the only money lost is the price paid for the option. The mathematical question is to determine that price.

First mathematical model of the markets was created by Bachelier in the 1900, by using the process known as *Brownian Motion*.

But Bachelier did not discover the price for the option.

Myron Scholes and Fischer Black came up with the option pricing formula (the Black-Scholes formula) in 1973. Many other people contributed to its development, especially Merton.

Merton R.C. and Scholes M.S. received the Nobel prize in economics in 1997, F. Black died (mid-fifties) in 1995.

2. Two real recent examples.

Collapse of Barings Bank.

This case demonstrates the leverage of derivatives, which allow to control the capital many times (10 times or more) over the initial capital.

Barings, Britain's oldest (200 year-old) merchant bank collapsed in 1995 because it could not meet the enormous trading obligations, which its trader (Nick Leeson) established in the name of the bank.

Barings had outstanding *futures* positions on Japanese SX and interest rates of $US\$27billion$, and debt on options on Nikkei of $US\$6.68b$.

Amazing when compared with the bank's capital of $615 million.

Long Term Capital Management (LTCM)

"This is the extraordinary story of a beautiful mathematical formula that changed the world, the financial markets, and indeed capitalism itself. It could do the unthinkable - it took the risk out of playing the money-markets.

To its inventors it brought the Nobel Prize for economics. To those who used it, it brought great wealth. But this glittering tale would end in tragedy." The Midas formula, BBC 1999.

Scholes and Merton with the dealers on Wall Street, started the Long Term Capital Management (LTCM) in 1996.

In first two years returns were 43% and 41% with the capital of $7 billion.

But in the third year the company had huge debts $100 billion, losing over $500 ml. on a single day.

It had to be bailed out by the US Reserve Banks and a consortium of 14 major international banks, for the fear of the collapse of the whole global financial system.

The Black-Scholes formula is used by traders who use their own modifications and adjustments.

This formula has changed the way we look at risk.

It is used to measure and trade some of the market volatility (the implied volatility).

Unlike in the case of Barings Bank (Leeson went to gaol), LTCM did not do illegal things.

Scholes is the managing partner in the Oak Hill Platinum multi-billion dollar hedge fund, and delivered a lecture at the Mathematical Fields Institute in Jan. 2003.

3. Models

If a model for stock prices is assumed then there is a rational price for the options.

Example: Assume the following model for evolution of stock price S_T

$$\text{Current Price} = \quad 10 \quad \begin{array}{c} 12 \\ \\ 8 \end{array}$$

Suppose interest rate is 10%.

Consider pricing of an option to buy the stock for \$10 at time T

Suppose the option is priced at $1 per share.

Consider the strategy: buy 200 options and sell 100 shares (selling what you don't have is allowed)

		$S_T = 12$	$S_T = 8$
Buy option on 200sh	-200	400	0
Sell (short) 100 shares	1000	-1200	-800
Invest	800	880	880
Profit	0	+80	+80

In either case a profit of $80 is realized.

This is an **arbitrage**, i.e. a strategy of making money with no risk involved.

Since such strategy exists more and more investors will buy the option. This will force the price of the call to go up.

Suppose the option is priced at \$2. Then the opposite strategy will give arbitrage: Sell options on 200 shares and buy 100 shares

		$S_T = 12$	$S_T = 8$
Sell calls on 200 sh	400	-400	0
Buy 100 shares	-1000	1200	800
Borrow	600	-660	-660
Profit	0	+140	+140

In this case the reverse strategy gives an arbitrage opportunity. Thus more and more people will sell options, which will push the price down.

Thus one needs to determine the price that would not allow for arbitrage strategies.

The arbitrage-free price for the option turns out to be \$1.36

Brownian Motion

Botanist R. Brown described the motion of a pollen particle suspended in fluid in 1828. It was observed that a particle moved in an irregular, random fashion.

In 1900 L. Bachelier used the Brownian motion as a model for movement of stock prices in his mathematical theory of speculation.

A. Einstein in 1905 explained Brownian motion as a result of molecular bombardment by the molecules of the fluid.

Mathematical foundation for Brownian motion as a stochastic process was done by N. Wiener in 1931, hence the Wiener process.

A. Einstein in 1905 (On the movement of small particles suspended in a stationary liquid demanded by the molecular-kinetic theory of heat. *Ann. Physik* 17, 549-560) obtained the equations for Brownian motion (from his theory Avogardo's number was also estimated) .

$$\frac{\partial p}{\partial t} = \frac{1}{2} c \frac{\partial^2 p}{\partial y^2},$$

where $p_t(x, y)$ is the probability density of a Brownian particle being at the position y at time t, when started at x at time 0.

If B_t denotes the position of a Brownian particle at time t, then the displacement $B_t - B_0$ is the effect of the purely random bombardment during time t.

Defining Properties of Brownian Motion.

$B_t - B_s$ is independent of the past, has Normal distribution with mean 0 and variance $t - s$, and B_t is a continuous functions of t.

Black-Scholes-Merton Model is based on Brownian motion.

Growth at a constant rate μ is described by the equation

$$\frac{dS_t}{S_t} = \mu dt$$

Growth rate on shares is not certain and modelled with the help of White Noise ξ_t derivative of Brownian motion $\mu + \sigma\xi_t$

$$\frac{dS_t}{S_t} = (\mu + \sigma\xi_t)dt = \mu dt + \sigma dB_t.$$

The problem is that the Brownian motion is not differentiable.

The meaning to the above equation is given by defining a new integral (Itô) and using integral equation

$$S_t = S_0 + \int_0^t \mu S_u du + \int_0^t \sigma S_u dB_u,$$

where the second integral is the new stochastic integral.

The solution of the Black-Scholes equation is given by is given by

$$S_t = S_0 e^{(\mu - \frac{1}{2}\sigma^2)t + \sigma B_t},$$

known as a Geometric Brownian motion.

Notice that it a bit different to what we would expect in the exponential, $(\mu - \frac{1}{2}\sigma^2)$ instead of μ. This is due to some new rules for stochastic integration (Itô's formula).

Chain Rule: Ito's formula for $f(B_t)$.

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f^{''}(B_t)dt$$

For example, take $f(x) = x^2$. Then $f'(x) = 2x$, $f^{''}(x) = 2$.
$d(B_t^2) = 2B_t dB_t + dt$,

which must be understood in the integral form

$$B_t^2 = 2\int_0^t B_u dB_u + t$$

or

$$\int_0^t B_u dB_u = \frac{1}{2}B_t^2 - \frac{1}{2}t.$$

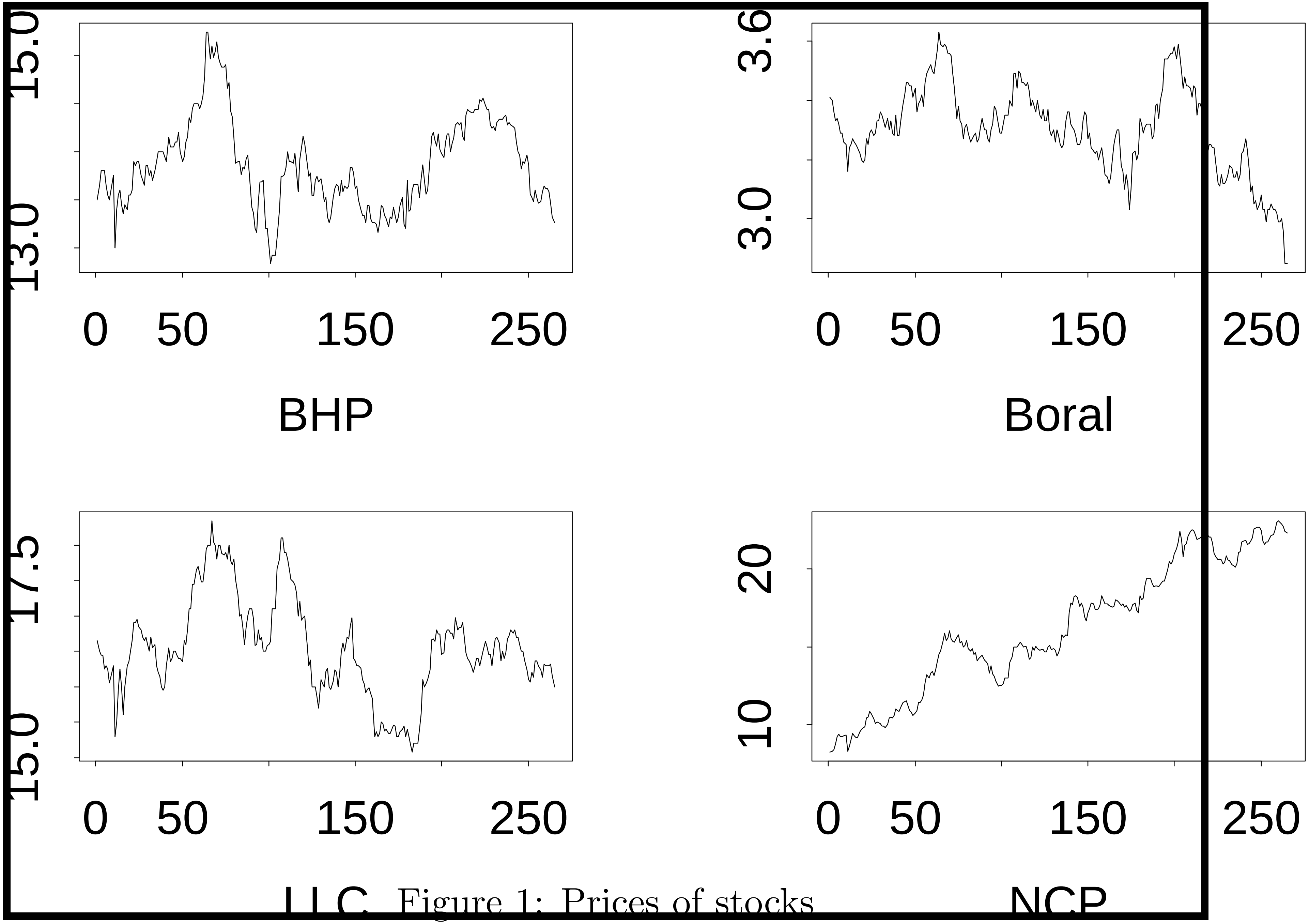

Figure 1: Prices of stocks

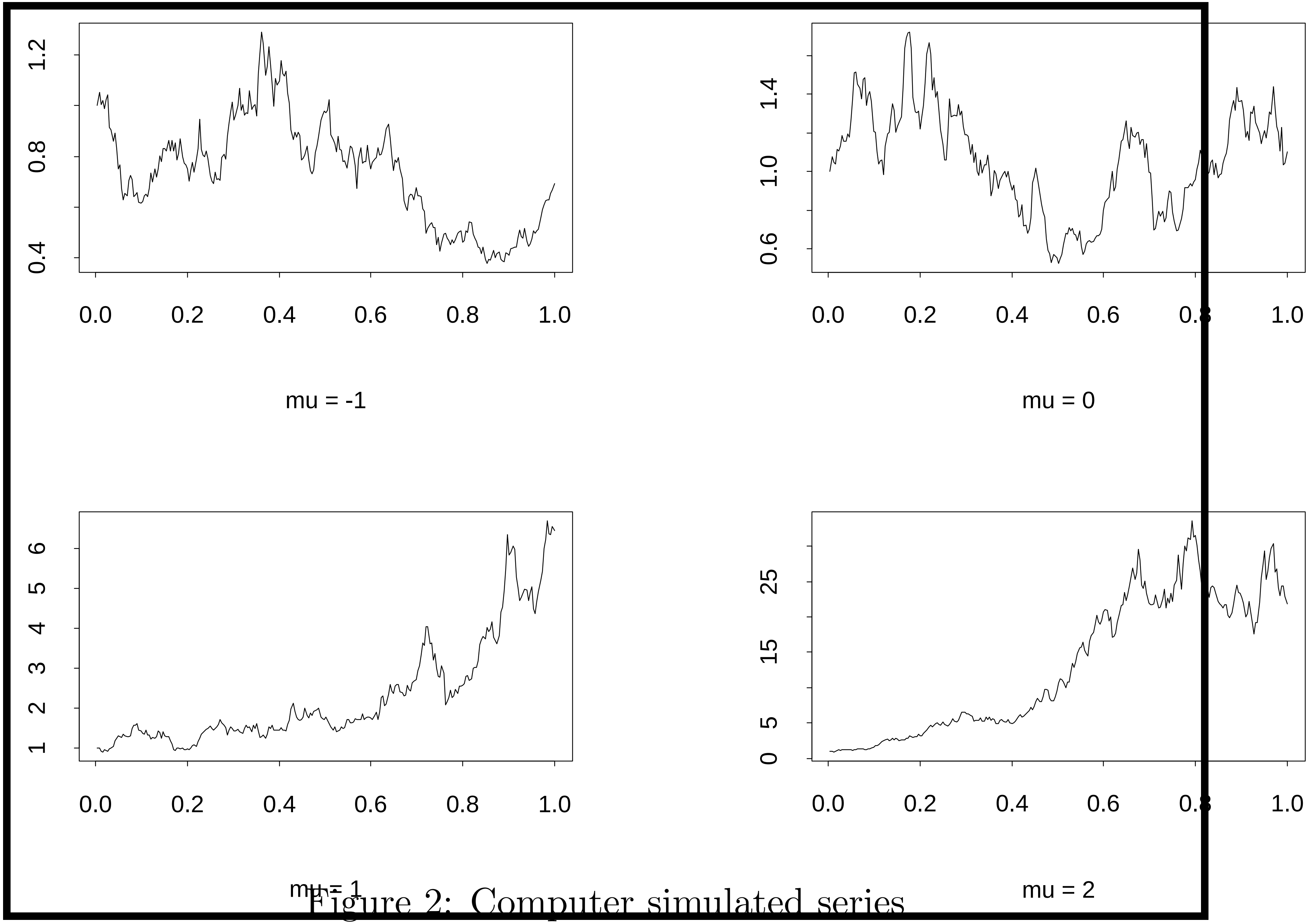

Figure 2: Computer simulated series

Option pricing in the Black-Scholes model

There are a number of derivations, we present the one from financial mathematical.

Idea: match the payoff of the option by a portfolio of tradables consisting of a_t shares and b_t cash (savings account).

Then the value of such a portfolio is

$$V_t = a_t S_t + b_t e^{rt}.$$

It should be self-financing,

$$dV_t = a_t dS_t + b_t d(e^{rt}),$$

and on maturity/expiration it should match the payoff of the option $V_T = C_T = g(S_T) = \max(S_T - K, 0)$. To avoid arbitrage, the value of the option at any time t is then given by the value of replicating portfolio $C_t = V_t$.

These equations plus Ito's formula imply the PDE for the price of the option, as well as coefficients in the replicating portfolio.

By Ito's formula (for a function of two variables) $C_t = C(S_t, t)$

$$dC_t = \frac{\partial C}{\partial x} dS_t + \frac{\partial C}{\partial t} dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial x^2} dt$$

On the other hand

$$dC_t = a_t dS_t + b_t d(e^{rt}).$$

Equating rhs' $a_t = \frac{\partial C}{\partial x}$, and $re^{rt} b_t = \frac{\partial C}{\partial t} dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial x^2}$

But $C_t = V_t = a_t S_t + b_t e^{rt}$, so we have

$$C_t = S_t \frac{\partial C}{\partial x} + \frac{1}{r}\Big(\frac{\partial C}{\partial t} dt + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 C}{\partial x^2}\Big)$$

Finally, replacing S_t by x, we obtain the Black-Scholes PDE

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 C}{\partial x^2} + rx\frac{\partial C}{\partial x} + \frac{\partial C}{\partial t} - rC = 0 \qquad (1)$$

A boundary condition is given by the value of option at maturity $C(x,T) = g(x) = (x-K)^+$.

Solution gives the Black-Scholes formula

$$C_t = S_t\Phi(h_t) - Ke^{-r(T-t)}\Phi(h_t - \sigma\sqrt{T-t}),$$

$$h_t = \frac{\ln(S_t/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

where $\Phi(a) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi}} e^{-v^2/2} dv$.

This PDE can be solved by using transformations. It is transformed into a heat (diffusion) equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

with appropriate boundary condition $u(x, 0) = u_0(x)$. Solution to the heat equation is well-known, we need to integrate the initial data against the normal density

$u(x, t) = \frac{1}{\sqrt{2\pi t}} \int u_0(y) e^{-(x-y)^2/4t} dy.$

Evaluation of the integral gives the Black-Scholes formula.

$C(x, t) = x\Phi(h_t) - Ke^{-r(T-t)}\Phi(h_t - \sigma\sqrt{T-t}),$
$h_t = \frac{\ln(x/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$

General Theory

The following questions are of interest in a general market model.

1. How can one tell that a market model does not admit arbitrage strategies? An answer provides conditions for no-arbitrage, the First Fundamental Theorem.

2. If the model is arbitrage-free, how to tell if it is possible to replicate an option by a self-financing portfolio?

An answer is provided by the Second Fundamental Theorem.

3. As a corollary, a general pricing formula is obtained.

These general principles apply not only for options on shares but also for capping interest rates payments on loans and FX options.

A general mathematical model specifies S_t^i, $i = 0, 1, 2, \ldots, m$ often described by (stochastic) equations.

One of them is assumed strictly positive and chosen as a numeraire, usually, the savings account, or the bond is a numeraire, $S_t^0 = \beta_t$.

In simple models, where time is discrete and the possible values of stocks are discrete,

Theorem. 1 (Fist Fundamental Theorem of asset pricing) *A market model does not admit arbitrage if there exists an equivalent martingale probability measure, such that makes S_t/β_t into a martingale.*

If a model admits arbitrage, then the following theory does not apply.

If a model does not admit arbitrage, then we can price options by no-arbitrage arguments, provided options can be replicated.

Theorem. 2 (Second Fundamental Theorem of asset pricing) *Any claim can be replicated by a self-financing portfolio if the equivalent martingale probability measure, such that makes S_t/β_t into a martingale is unique. (also known as market completeness)*

Corollary. *The price of option paying X at time T is given by*

$$C_t = E(\frac{X\beta_t}{\beta_T}|S_{[0,t]}).$$

More often than not this formula does not compute.

Impled volatility

As many options are also traded on options exchange, their market prices are observed.

The volatility parameter σ is the only unknown parameter not observed directly.

Under the BS model σ is the standard deviation of returns

$$\ln \frac{S(t_{i+1})}{S(t_i)}$$

However, in many cases this volatility, called the historic volatility, does not return the observed options prices.

Practioners often use the implied volatility, obtained by solving the BS equation for σ from the observed C_t's.

Models are developed to explain the observed vols/ prices.

Contracts using volatility are traded.

Other applications

Options methodology is being applied in various markets, such as electricity markets, and weather derivatives.

There are unsolved problems in interest rates markets, such as pricing of American Put (related to optimal stopping of diffusions), pricing of exotic options (barrier, knock in, knock out, asian-average strike or average rate, etc.), Bermuda swaptions (options to exchange floating rates payments to fixed rates, exercised only few times a year), Credit Risk (pricing and hedging of defaultable bonds), Real Options (options on projects)

Electricity and weather derivatives.

These problems require mathematical modelling and analysis.